STATE SPACE ANALYSIS

Introduction

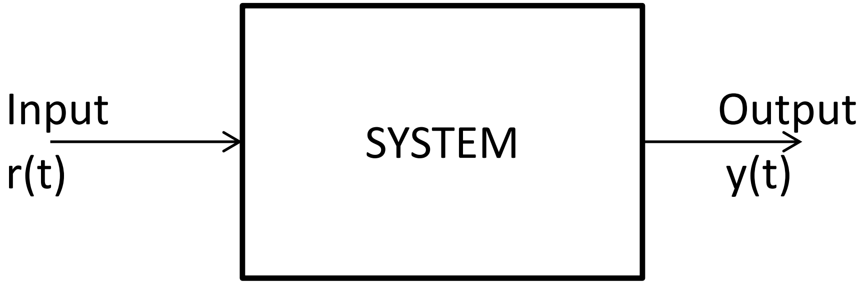
- State space analysis is an powerful and modern approach for the design and analysis of control systems.
- The conventional or old methods for the design and analysis of control systems is based on transfer function method.
- The transfer function method for design and analysis had many drawbacks such as..
  - Transfer function is defined under zero initial conditions
  - Applicable to LTI systems
  - SISO systems
  - Does not provide the information regarding internal state of the system
- Initial conditions can be incorporated in the system design
- State equations are highly compatible for simulation on analog or digital computers

Advantages of state variable analysis.

This can be applicable to

- Linear systems
- Non-linear system
- Time variant systems
- Time invariant systems
- Multiple input multiple output systems
- This gives idea about the internal state of the system

Concept



The output not only depends on the input applied to the system for $t > t_0$, but also on the initial conditions at time $t = t_0$

$$y(t) = y(t)\big|_{t=t0} + y(t)\big|_{t\geq t0}$$

$$= \int_{-\infty}^{t_0} y(t) + \int_{t_0}^{t} y(t)$$

$$= y(t_0) + \int_{0}^{t} y(t)$$

The term $y(t_0)$ is called the state of the system.

The variable that represents this state of the system is called the state variable

Definitions

State: The state of a dynamic system is the smallest set of variables called state variables such that the knowledge of these variables at time t = t_o (Initial condition), together with the knowledge of input for t ≥ t_0 , completely determines the behavior of the system for any time $t \geq t_0$.

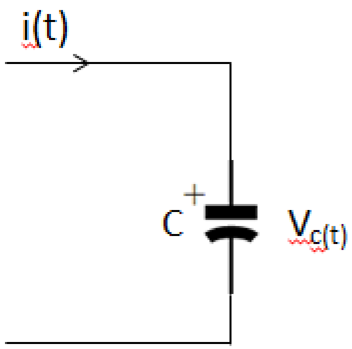
State Variables: A set of variables which describes the system at any time instant are called state variables

State vector: A vector whose elements are the state variables

State space: The n-dimensional space whose co-ordinate axes consists of the x_1 axis, x_2 axis,.... x_n axis, (where x_1 , x_2 ,..... x_n are state variables:) is called a state space.

Illustration

Consider the circuit shown in figure



$$i(t) = C\,\frac{dv_c(t)}{dt}$$

$$V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

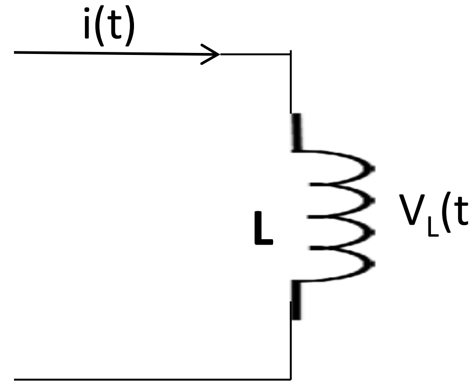
$$= \frac{1}{C}\int_{-\infty}^{t_0} i(t)dt + \frac{1}{C}\int_{t_0}^{t} i(t)dt$$

$$= V_c(t_0) + V_c(t)$$

$V_c(t_0)$ = initial voltage across capacitor

The voltage across capacitor can be taken as a state variable

Consider the circuit shown in figure



$$V_L(t) = L\,\frac{di(t)}{dt}$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} V_L(t)dt$$

$$= \frac{1}{L}\int_{-\infty}^{t_0} V_L(t)\,dt + \frac{1}{L}\int_{t_0}^{t} V_L(t)\,dt$$

$$= V_L(t_0) + V_L(t)$$

$V_L(t_0)$ = initial voltage across Inductor

The current through inductor can be taken as a state variable

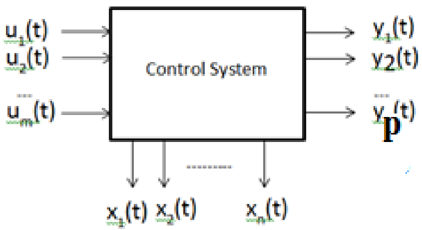
State Model

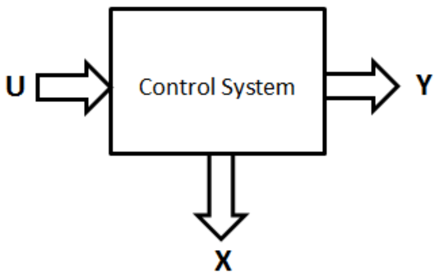
consider a multi-input & multi-output system is having

m inputs $u_1(t)$, $u_2(t)$,…….$u_m(t)$

p outputs $y_1(t)$, $y_2(t)$,…….$y_p(t)$

n state variables $x_1(t)$, $x_2(t)$,…….$x_n(t)$





The different variables may be represented by the vectors as shown below

Input vector U(t) = $\begin{bmatrix} u_1(t) \\ u_2(t) \\ .. \\ u_m(t) \end{bmatrix}$; Output vector Y(t) = $\begin{bmatrix} y_1(t) \\ y_2(t) \\ .. \\ y_p(t) \end{bmatrix}$

State variable vector X(t) = $\begin{bmatrix} x_1(t) \\ x_2(t) \\ .. \\ x_n(t) \end{bmatrix}$

State Equations

The state variable representation can be arranged in the form of n number of first order differential equations as shown below:

$$\frac{dx_1}{dt} = \dot{x}_1 = f1(x_1,x_2,....x_n ; u_1,u_2.....u_m)$$

$$\frac{dx_2}{dt} = \dot{x}_2 = f1(x_1,x_2,....x_n ; u_1,u_2.....u_m)$$

……

……

$$\frac{dx_n}{dt} = \dot{x}_n = f1(x_1,x_2,....x_n ; u_1,u_2.....u_m)$$

In vector notation, $\dot{X}(t)= f(X(t),U(t))$

Similarly the output vector $Y(t) = f(X(t),U(t))$

State Model of Linear System

The state model of a system consist of state equation and output equation.

The state equation of a system is a function of state variables and inputs.

For LTI systems, the first derivatives of state variables can be expressed as a linear combination of sate variables and inputs

$\dot{x}_1 = a_{11}x_1+a_{12}x_2+\text{----}+a_{1n}x_n + b_{11}u_1+b_{12}u_2+\text{---}+b_{1m}u_m$

$\dot{x}_2 = a_{21}x_1+a_{22}x_2+\text{----}+a_{2n}x_n + b_{21}u_1+b_{22}u_2+\text{---}+b_{2m}u_m$

$\dot{x}_n = a_{n1}x_1+a_{n2}x_2+\text{----}+a_{nn}x_n + b_{n1}u_1+b_{n2}u_2+\text{---}+b_{nm}u_m$

where the coefficients a_{ij} and b_{ij} are constants

In the matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ .. \\ .. \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & & b_{1m} \\ b_{21} & \cdots & b_{2m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ .. \\ .. \\ u_m \end{bmatrix}$$

$\dot{X}(t) = A\,X(t) + B\,U(t)$……… state equation

where, A is state matrix of size (n×n)

B is the input matrix of size (n×m)

X(t) is the state vector of size (n×1)

U(t) is the input vector of size (m×1)

Output equation

The output at any time are functions of state variables and inputs.

output vector, $Y(t) = f(x(t), U(t))$

Hence the output variables can be expressed as a linear combination of state variables and inputs.

$y_1 = c_{11}x_1 + c_{12}x_2 + \text{----} + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \text{---} + d_{1m}u_m$

$y_2 = c_{21}x_1 + c_{22}x_2 + \text{----} + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \text{---} + d_{2m}u_m$

$y_p = c_{p1}x_1 + c_{p2}x_2 + \text{----} + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \text{---} + d_{pm}u_m$

where the coefficients c_{ij} and d_{ij} are constants

In the matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot\cdot \\ \cdot\cdot \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ c_{21} & \cdots & c_{2n} \\ \vdots & \ddots & \vdots \\ c_{p1} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot\cdot \\ \cdot\cdot \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & & d_{1m} \\ d_{21} & \cdots & d_{2m} \\ \vdots & \ddots & \vdots \\ d_{p1} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \cdot\cdot \\ \cdot\cdot \\ u_m \end{bmatrix}$$

Y(t) = C X(t) + D U(t)………….. output equation

where, C is the output matrix of size (p×n)

D is the transmission matrix of size (p×m)

X(t) is the state vector of size (n×1)

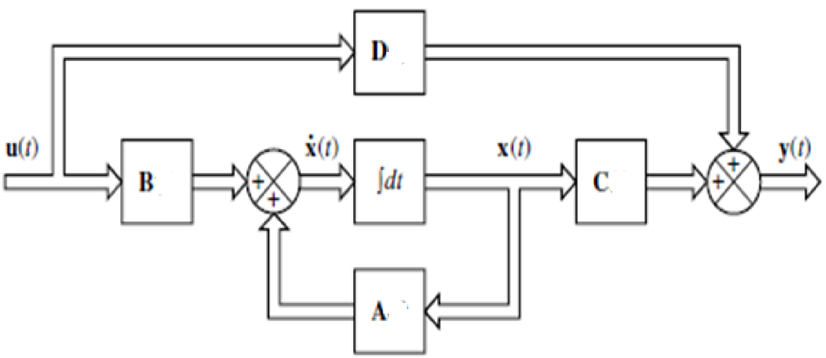
Y(t) is the output vector of size (p×1)

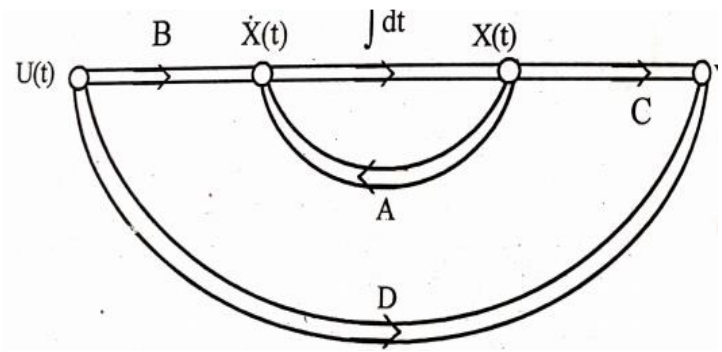
U(t) is the input vector of size (m×1)

State Model

$\dot{X}(t) = A\,X(t) + B\,U(t)$ state equation

$Y(t) = C\,X(t) + D\,U(t)$ output equation





Selection of state variables

- The state variables of a system are not unique.
- There are many choices for a given system

Guide lines:

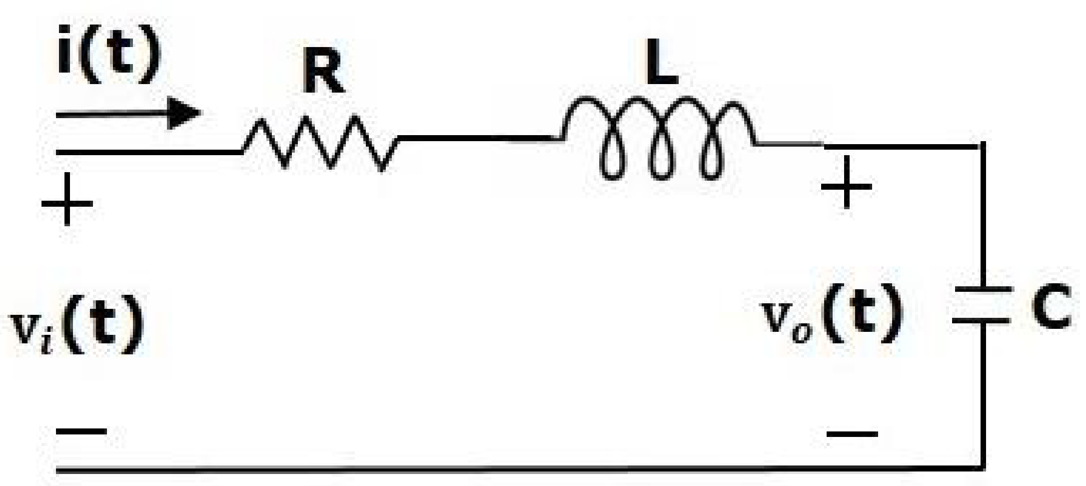
1. For a physical systems, the number of state variables needed to represent the system must be equal to the number of energy storing elements present in the system
2. If a system is represented by a linear constant coefficient differential equation, then the number of state variables needed to represent the system must be equal to the order of the differential equation
3. If a system is represented by a transfer function, then the number of sate variables needed to represent the system must be equal to the highest power of s in the denominator of the transfer function.

State space Representation using Physical variables

- In state-space modeling of the systems, the choice of sate variables is arbitrary.
- One of the possible choice is physical variables.
- The state equations are obtained from the differential equations governing the system

State Space Model

Consider the following series of the RLC circuit. It is having an input voltage $v_i(t)$ and the current flowing through the circuit is i(t).



- There are two storage elements (inductor and capacitor) in this circuit. So, the number of the state variables is equal to two.
- These state variables are the current flowing through the inductor, $i(t)$ and the voltage across capacitor, $v_c(t)$.
- From the circuit, the output voltage, $v_0(t)$ is equal to the voltage across capacitor, $v_c(t)$.

$Y(t) = v_0(t) = v_c(t)$

Apply KVL around the loop,

$$V_i(t) = R\,i(t) + L\,\frac{di(t)}{dt} + v_c(t)$$

$$\dot{i(t)} = \frac{di(t)}{dt} = -\frac{R}{L}\,i(t) - \frac{1}{L}\,v_c(t) + \frac{1}{L}\,V_i(t)$$

The voltage across the capacitor is

$$v_c(t) = \frac{1}{c}\int i(t)dt$$

Differentiate the equation with respect to time,

$$\dot{v_c(t)} = \frac{dv_c(t)}{dt} = \frac{i(t)}{c}$$

State vector, $X = \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$; Differential state vector, $\dot{X} = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{dv_c(t)}{dt} \end{bmatrix}$

Arrange the differential equations and output equation into standard form of state space model as,

$$\dot{X} = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{dv_c(t)}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [v_i(t)]$$

$$Y = [0 \quad 1] \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

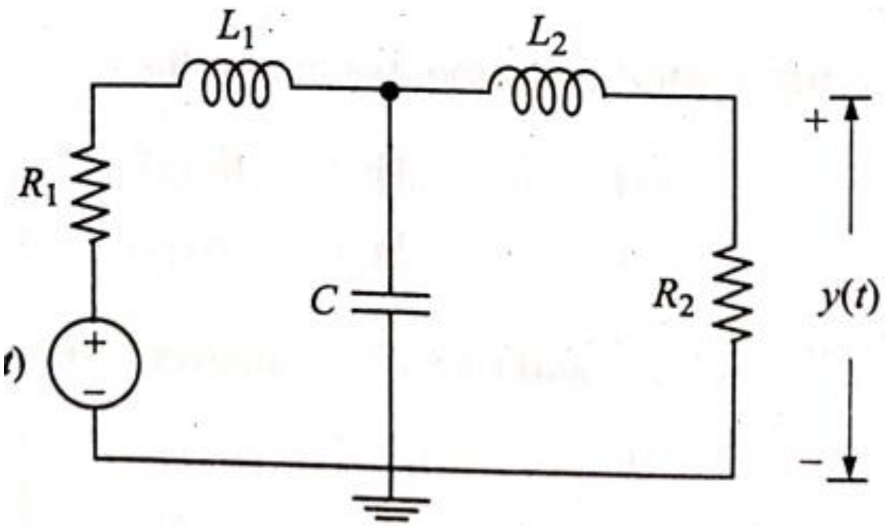
$$\dot{X}(t) = A\,X(t) + B\,U(t)$$

$$Y(t) = C\,X(t) + D\,U(t)$$

Here $A = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}$; $B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$; $C = [0 \quad 1]$; $D = [0]$

Problem

Represent the electrical circuit shown by a state model



Solution

Since there are three energy storing elements, choose three state variables to represent the systems

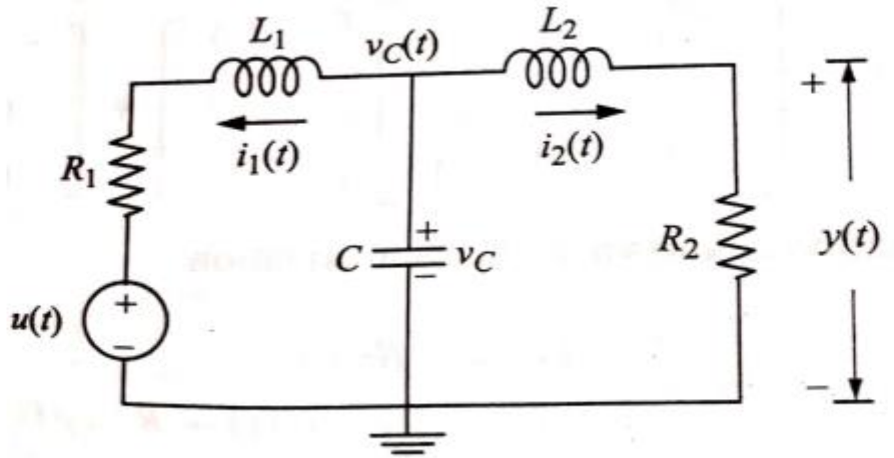
The current through the inductors i_1,i_2 and voltage across the capacitor v_c are taken as state variables

Let the three sate variables be x1, x2 and x3 be related to physical quantities as shown

Let, $i_1 = x_1$,

$i_2 = x_2$,

$v_c = x_3$



Applying KVL to loop 1,

$$L_1 \frac{di_1(t)}{dt} + R_1 i_1(t) + u(t) - v_c(t) = 0$$

$$\Longrightarrow \quad L_1 \frac{dx_1(t)}{dt} + R_1 x_1(t) + u(t) - x_3(t) = 0$$

$$\Longrightarrow \quad L_1 \dot{x}_1(t) + R_1 x_1(t) + u(t) - x_3(t) = 0$$

$$\Longrightarrow \quad \dot{x}_1(t) = -\frac{R_1}{L_1} x_1(t) + \frac{1}{L_1} x_3(t) - \frac{1}{L_1} u(t) \text{-------} (1)$$

Applying KVL to loop 2,

$$L_2 \frac{di_2(t)}{dt} + R_2 i_2(t) - v_c(t) = 0$$

$$\Longrightarrow \quad L_2 \frac{dx_2(t)}{dt} + R_2 x_2(t) - x_3(t) = 0$$

$$\Longrightarrow \quad L_2 \, \dot{x}_2(t) + R_2 x_2(t) - x_3(t) = 0$$

$$\Longrightarrow \quad \dot{x}_2(t) = -\frac{R_2}{L_2} x_2(t) + \frac{1}{L_2} x_3(t) \text{------------------} (2)$$

Applying KCL at node $v_c(t)$,

$$i_1(t) + i_2(t) + C\frac{dv_c(t)}{dt} = 0$$

$$\Longrightarrow \quad x_1(t) + x_2(t) + C\frac{dx_3(t)}{dt} = 0$$

$$\Longrightarrow \quad \dot{x}_3(t) = -\frac{1}{C}x_1(t) - \frac{1}{C}x_2(t) \text{ ---------------(3)}$$

Putting 1, 2and 3 in matrix form,

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix} = \begin{bmatrix}-\frac{R_1}{L_1} & 0 & \frac{1}{L_1}\\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2}\\ -\frac{1}{C} & -\frac{1}{C} & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}-\frac{1}{L_1}\\ 0\\ 0\end{bmatrix}[u(t)]$$

This is State Equation

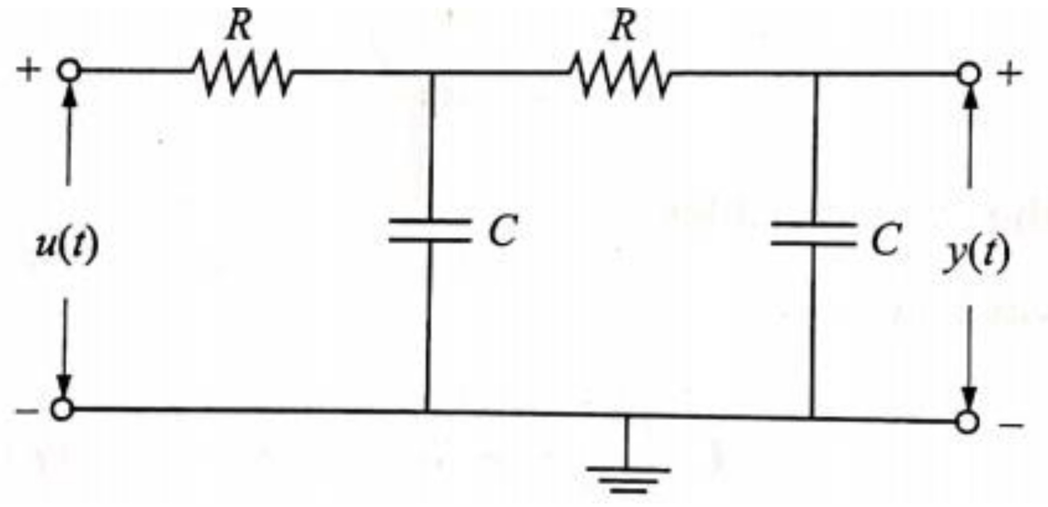
$$y(t) = R_2 i_2(t) = R_2 x_2(t)$$

$$y(t) = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

This is output equation

Problem

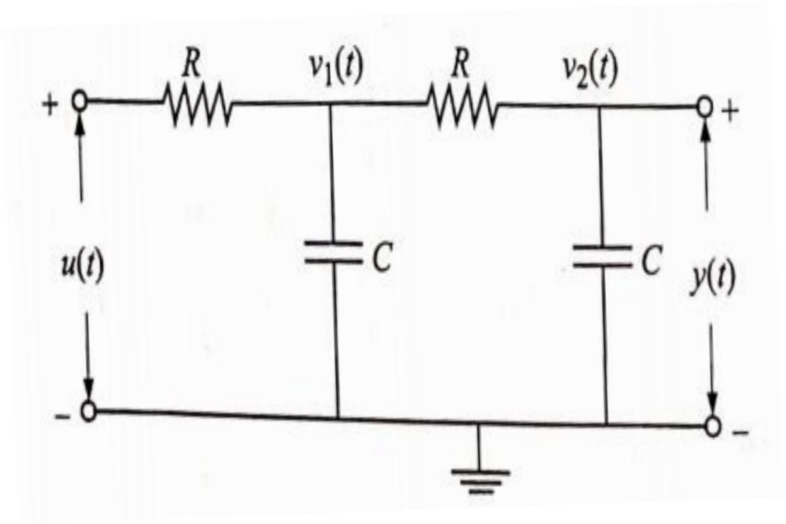
Obtain the state model for a system represented by an electrical system as shown in figure



Solution

Since there are two energy storage elements present in the system, assume two state variables to describe the system behavior.

Let the two state variables be x_1 and x_2 be related to physical quantities as shown



Let $v_1(t) = x_1(t)$

$v_2(t) = x_2(t)$

Applying KCL at node $v_1(t)$,

$$\frac{v_1(t)-u(t)}{R} + C\frac{dv_1(t)}{dt} + \frac{v_1(t)-v_2(t)}{R} = 0$$

$$\Longrightarrow \quad \frac{x_1(t)-u(t)}{R} + C\frac{dx_1(t)}{dt} + \frac{x_1(t)-x_2(t)}{R} = 0$$

$$\Longrightarrow \quad \frac{2x_1(t)}{R} - \frac{u(t)}{R} + C\,\dot{x}_1(t) - \frac{x_2(t)}{R} = 0$$

$$\Longrightarrow \quad \dot{x}_1(t) = -\frac{2x_1(t)}{RC} + \frac{x_2(t)}{RC} + \frac{u(t)}{RC} \text{ ------------ } (1)$$

Applying KCL at node $v_2(t)$,

$$C\,\frac{dv_2(t)}{dt} + \frac{v_2(t)-v_1(t)}{R} = 0$$

$$\Longrightarrow \quad C\,\frac{dx_2(t)}{dt} + \frac{x_2(t)-x_1(t)}{R} = 0$$

$$\Longrightarrow \quad C\,\dot{x}_2(t) - \frac{x_1(t)}{R} + \frac{x_2(t)}{R} = 0$$

$$\Longrightarrow \quad \dot{x}_2(t) = \frac{x_1(t)}{RC} - \frac{x_2(t)}{RC} \qquad \text{----------(2)}$$

putting 1 and 2 in matrix form,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{-2}{RC} & \frac{1}{RC} \\ \frac{1}{RC} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} [u(t)]$$

This is the state equation

The output of the circuit is given by

$$y(t) = v_2(t)$$

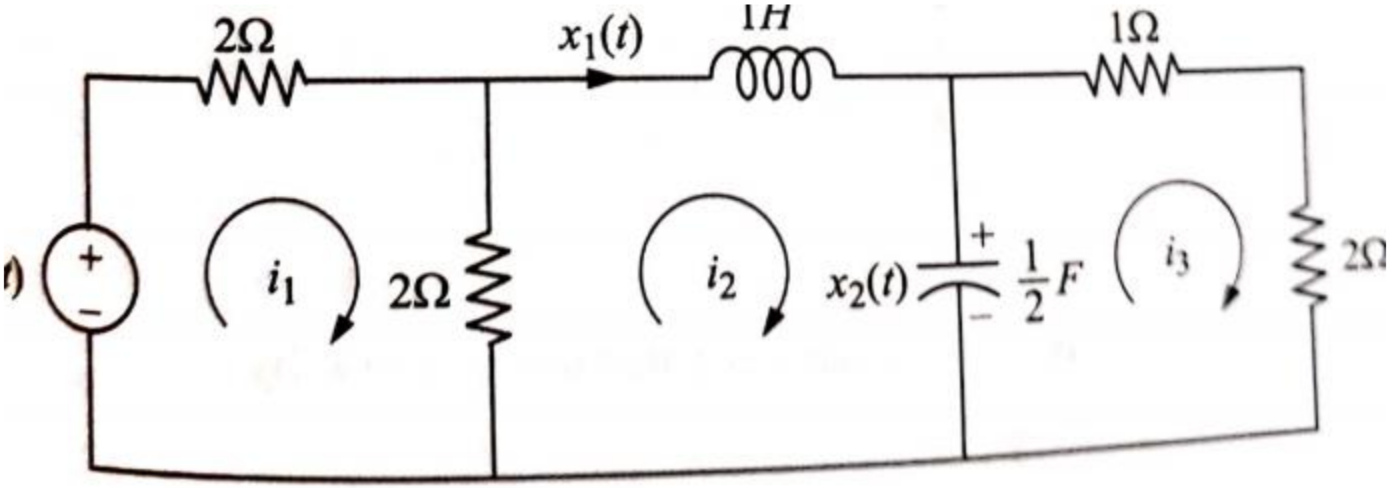
$$= x_2(t)$$

$$= [0 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

This is the output equation

Problem

Represent the electrical network by a state equation



State representation using Phase variables

- The phase variables are defined as those particular state variables which are obtained from one of the system variables and its derivatives.
- Usually the variables used is the system output and the remaining state variables are then derivatives of the output.
- The state model using phase variables can be easily determined if the system model is already known in the differential equation or transfer function form.

Consider the following nth order linear differential equation relating the output y(t) to the input u(t) of a system.

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + a_2 \frac{d^{n-2} y}{dt^{n-2}} + \text{-----} + a_{n-1} \frac{dy}{dt} + a_n y = u$$

Let us define the state variables as

$$x_1 = y$$

$$x_2 = \frac{dy}{dt} = \frac{dx_1}{dt}$$

$$x_3 = \frac{d^2 y}{dt^2} = \frac{d\dot{y}}{dt} = \frac{dx_2}{dt}$$

$$\vdots \; \vdots \; \vdots$$

$$x_n = \frac{d^{n-1} y}{dt^{n-1}} = \frac{dx_{n-1}}{dt}$$

From the above equations we can write

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\vdots \; \vdots$$

$$\dot{x}_{n-1} = x_n$$

$$\dot{x}_n + a_1 x_n + \ldots.. + a_{n-1} x_2 + a_n x_1 = u$$

$$\dot{x}_n = - a_n x_1 - a_{n-1} x_2 \ldots\ldots.. - a_1 x_n + u$$

writing the above state equation in vector matrix form,

$$\dot{X}(t) = AX(t) + Bu(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & & 0 \\ \vdots & \ddots & \vdots \\ -an & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \\ \dot{x}_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ . \\ 1 \end{bmatrix} [u]$$

Output equation can be written as

$$Y(t) = C\,X(t) = [1\ 0\ 0\ ..] \begin{bmatrix} x_1 \\ x_2 \\ \\ \dot{x}_n \end{bmatrix}$$

Problem

Construct a state model for a system characterized by the differential equation,

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0$$

Also give the block diagram representation of the state model

Solution: Let us choose y and their derivatives as state variables. The system is governed by third order differential equation, so the number of state variables required are three.

Let the state variables x_1, x_2 and x_3 are related to phase variables as follows.

$$x_1 = y$$

$$x_2 = \frac{dy}{dt} = \dot{x}_1$$

$$x_3 = \frac{d^2y}{dt^2} = \frac{dx_2}{dt} = \dot{x}_2$$

Put $y = x_1$, $\frac{dy}{dt} = x_2$, $\frac{d^2y}{dt^2} = x_3$ and $\frac{d^3y}{dt^3} = \dot{x}_3$ in the given equation

$$\therefore \dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + u = 0$$

$$\Longrightarrow \dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u$$

The state equations are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 - u$$

Arranging the state equations in the matrix form,

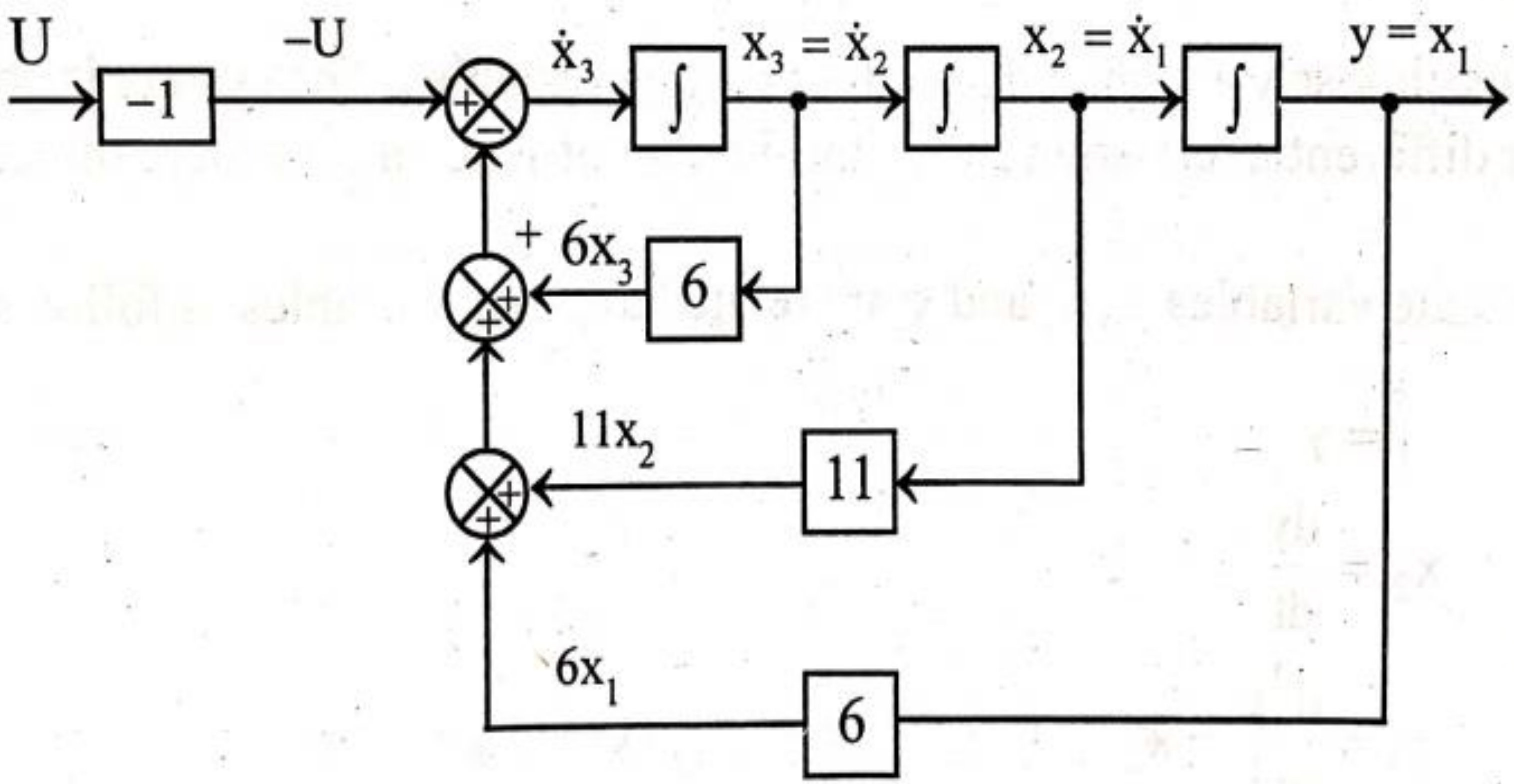
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [u]$$

Here y = output

But $y = x_1$

$\therefore$ The output equation is , $y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

The block diagram for the state model is



Problem

Represent the differential equation given below in a state model

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 6\,\frac{dy}{dt} + 7y = 2u(t)$$

Solution:

Since, the given equation is a third-order differential equation, choose three state variables to represent the system

Let $y(t) = x_1(t)$

$\dot{y}(t) = x_2(t)$

$\ddot{y}(t) = x_3(t)$

where $x_1(t)$, $x_2(t)$, $x_3(t)$ are the state variables of the system.

$$y(t) = x_1(t)$$

$$\dot{y}(t) = x_2(t) = \dot{x}_1(t) \qquad \text{-----------(1)}$$

$$\ddot{y}(t) = x_3(t) = \dot{x}_2(t) \qquad \text{-----------(2)}$$

From the given diff. equation,

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + 6\,\frac{dy}{dt} + 7y = 2u(t)$$

$$\dddot{y}(t) + \ddot{y}(t) + 6\dot{y}(t) + 7y = 2u(t)$$

$$\dot{x}_3(t) + x_3(t) + 6\,x_2(t) + 7\,x_1(t) = 2u(t)$$

$$\dot{x}_3(t) = -7\,x_1(t) - 6\,x_2(t) - x_3(t) + 2u(t) \quad \text{-------}(3)$$

$$\dot{x}_2(t) = x_3(t) \quad \text{------}(2)$$

$$\dot{x}_1(t) = x_2(t) \quad \text{------}(1)$$

Putting the above equations in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

This is the state equation

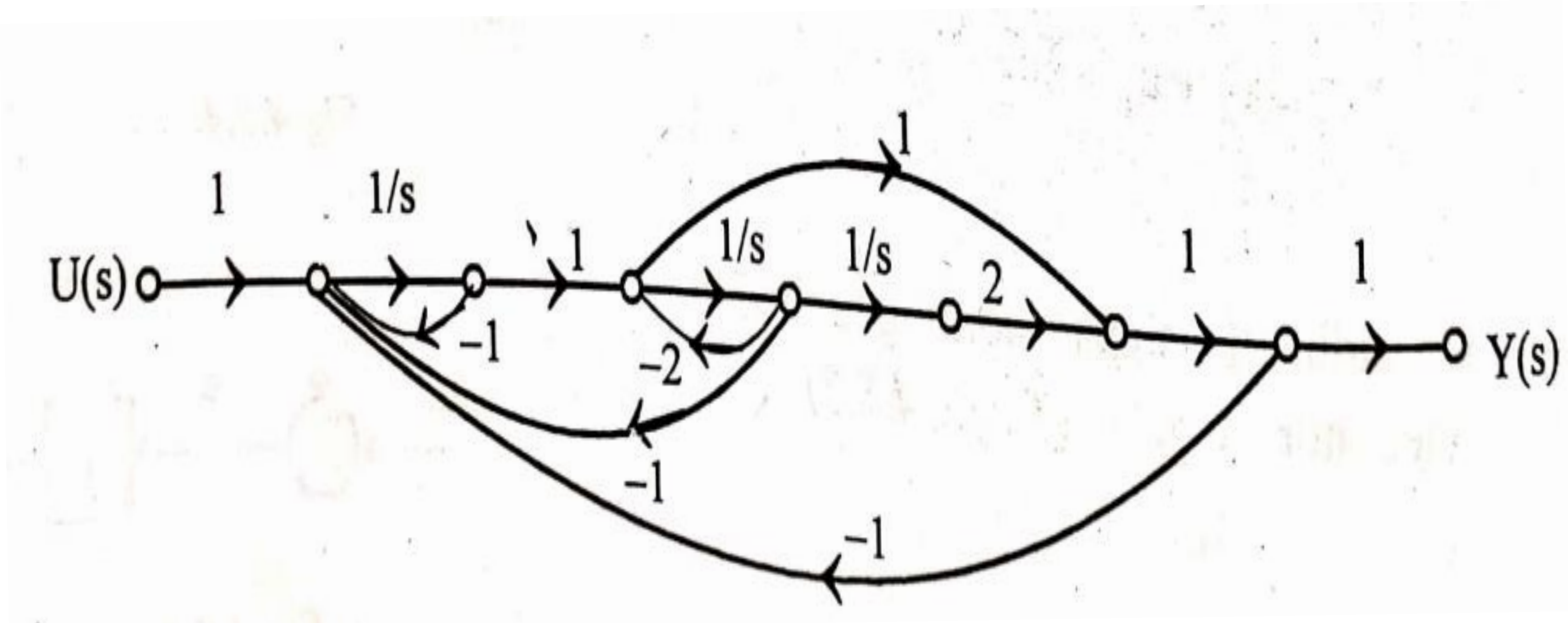
The output expression is $y(t) = x_1(t)$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This is the output equation

Problem

Obtain the state model for the signal flow graph given below:



Problem

Obtain the state model of the system whose transfer function is given by $\frac{Y(s)}{U(s)} = \frac{24}{s^3+9s^2+26s+24}$

Solution:

$$\frac{Y(s)}{U(s)} = \frac{24}{s^3+9s^2+26s+24}$$

Cross-multiplying yields

$$[s^3 + 9s^2 + 26s + 24]\, Y(s) = 24\, U(s)$$

$$s^3\, Y(s) + 9s^2\, Y(s) + 26sY(s) + 24\, Y(s) = 24U(s)$$

Taking inverse Laplace transforms,

$$\frac{d^3y(t)}{dt^3} + 9\frac{d^2y(t)}{dt^2} + 26\frac{dy(t)}{dt} + 24\, y(t) = 24u(t)$$

$$\dddot{y}(t) + 9\ddot{y}(t) + 26\dot{y}(t) + 24\, y(t) = 24u(t)$$

Choosing the state variables as successive derivatives

$x_1(t) = y(t)$

$x_2(t) = \dot{y}(t)$

$x_3(t) = \ddot{y}(t)$

$x_1(t) = y(t)$

$\dot{x}_1(t) = x_2(t) = \dot{y}(t)$ -----(1)

$\dot{x}_2(t) = x_3(t) = \ddot{y}(t)$ -----(2)

$\dot{x}_3(t) = \dddot{y}(t)$

$\dddot{y}(t) + 9\ddot{y}(t) + 26\dot{y}(t) + 24\, y(t) = 24u(t)$

$\dot{x}_3(t) + 9x_3(t) + 26x_2(t) + 24x_1(t) = 24u(t)$

$\dot{x}_3(t) = - 24x_1(t) - 26x_2(t) - 9x_3(t) + 24u(t)$------(3)

Putting equations 1, 2 and 3 in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} [u]$$

The output expression is $y(t) = x_1(t)$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matlab

Obtain the state model of the system whose transfer function is given by $\frac{\mathrm{Y(s)}}{\mathrm{U(s)}} = \frac{24}{s^3+9s^2+26s+24}$ using Matlab

```
>> num=[24];
>> den=[1 9 26 24];
>> [A,B,C,D] = tf2ss(num,den)
```

```
A =

    -9   -26   -24
     1     0     0
     0     1     0


B =

     1
     0
     0


C =

     0     0    24


D =

     0
```

Problem

Obtain the state model of the system whose transfer function is given by $\frac{Y(s)}{U(s)} = \frac{1}{s^2+s+1}$

Solution: $$\frac{Y(s)}{U(s)} = \frac{1}{s^2+s+1}$$

$$\Longrightarrow \quad (s^2 + s + 1)Y(s) = U(s)$$

$$s^2\,Y(s) + s\,Y(s) + Y(s) = U(s)$$

Taking inverse Laplace transform on both sides,

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} + y(t) = u(t)$$

Let $y(t) = x_1$

$$\frac{dy(t)}{dt} = x_2 = \dot{x}_1 \text{ and } u(t) = u$$

Then the state equation is, $\dot{x}_2 = -x_1 - x_2 + u$

The output equation is, $y(t) = y = x_1$

The state space model is

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Problem

Obtain the state model of the system whose transfer function is given by $\frac{s^2+7s+2}{s^3+9s^2+26s+24}$

Solution:

$$\frac{Y(s)}{U(s)} = \frac{s^2+7s+2}{s^3+9s^2+26s+24}$$

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{C(s)} \text{ x } \frac{C(s)}{U(s)}$$

$$\frac{Y(s)}{C(s)} = s^2 + 7s + 2 \qquad \text{--------(1)}$$

$$\frac{C(s)}{U(s)} = \frac{1}{s^3+9s^2+26s+24} \qquad \text{-------(2)}$$

Consider equation (2), $$\frac{C(s)}{U(s)} = \frac{1}{s^3+9s^2+26s+24}$$

Cross-multiplying on both sides,

$$[s^3 + 9s^2 + 26s + 24]\, C(s) = U(s)$$

$$s^3\, C(s) + 9s^2\, C(s) + 26s\, C(s) + 24\, C(s) = U(s)$$

Taking inverse Laplace transform,

$$\frac{d^3c(t)}{dt^3} + 9\,\frac{d^2c(t)}{dt^2} + 26\,\frac{dc(t)}{dt} + 24\, c(t) = u(t)$$

$$\dddot{c}(t) + 9\,\ddot{c}(t) + 26\,\dot{c}(t) + 24\, c(t) = u(t)$$

$$x_1(t) = c(t)$$

$$\dot{x}_1(t) = x_2(t) = \dot{c}(t) \qquad \text{-----(3)}$$

$$\dot{x}_2(t) = x_3(t) = \ddot{c}(t) \qquad \text{-----(4)}$$

$$\dot{x}_3(t) = \dddot{c}(t)$$

$$\dddot{c}(t) + 9\ddot{c}(t) + 26\dot{c}(t) + 24c(t) = u(t)$$

$$\dot{x}_3(t) + 9\, x_3(t) + 26x_2(t) + 24x_1(t) = u(t)$$

$$\dot{x}_3(t) = -\, 24x_1(t) - 26x_2(t) - 9\, x_3(t) + u(t) \quad \text{-----}(5)$$

Putting equations 3, 4 and 5 in matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [u]$$

Consider equation (1),

$$\frac{Y(s)}{C(s)} = s^2 + 7s + 2$$

$$Y(s) = [s^2 + 7s + 2]C(s)$$

$$Y(s) = s^2C(s) + 7sC(s) + 2C(s)$$

Taking inverse Laplace transform,

$$y(t) = \ddot{c}(t) + 7\dot{c}(t) + 2c(t)$$

$$y(t) = 2x_1(t) + 7x_2(t) + x_3(t)$$

$$y(t) = [2 \;\; 7 \;\; 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matlab

Obtain the state model of the system whose transfer function is given by $\frac{s^2+7s+2}{s^3+9s^2+26s+24}$ using Matlab

```
>> num=[1 7 2];
>> den=[1 9 26 24];
>> [A,B,C,D] = tf2ss(num,den)

A =

    -9   -26   -24
     1     0     0
     0     1     0


B =

     1
     0
     0


C =

     1     7     2


D =

     0
```

Problem

A feedback system has a closed-loop transfer function $\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)}$

Solution:

$$\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)}$$

By partial fraction expansion,

$$\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)} = \frac{A}{(s+2)} + \frac{B}{(s+3)} + \frac{C}{(s+4)}$$

Solving for A, B and C

A = 3; B = - 4; C = 1

$$\frac{Y(s)}{U(s)} = \frac{2(s+5)}{(s+2)(s+3)(s+4)} = \frac{3}{(s+2)} - \frac{4}{(s+3)} + \frac{1}{(s+4)} \text{ -----(1)}$$

$$= \frac{3}{s(1+2/s)} - \frac{4}{s(1+3/s)} + \frac{1}{s(1+4/s)}$$

$$= \frac{\frac{1}{s}}{(1+\frac{1}{s}*2)} \text{ x } 3 - \frac{\frac{1}{s}}{(1+\frac{1}{s}*3)} \text{ x } 4 + \frac{\frac{1}{s}}{(1+\frac{1}{s}*4)}$$

$$\therefore Y(s) = [\frac{\frac{1}{s}}{(1+\frac{1}{s}*2)} \text{ x } 3 - \frac{\frac{1}{s}}{(1+\frac{1}{s}*3)} \text{ x } 4 + \frac{\frac{1}{s}}{(1+\frac{1}{s}*4)}]u(s)$$

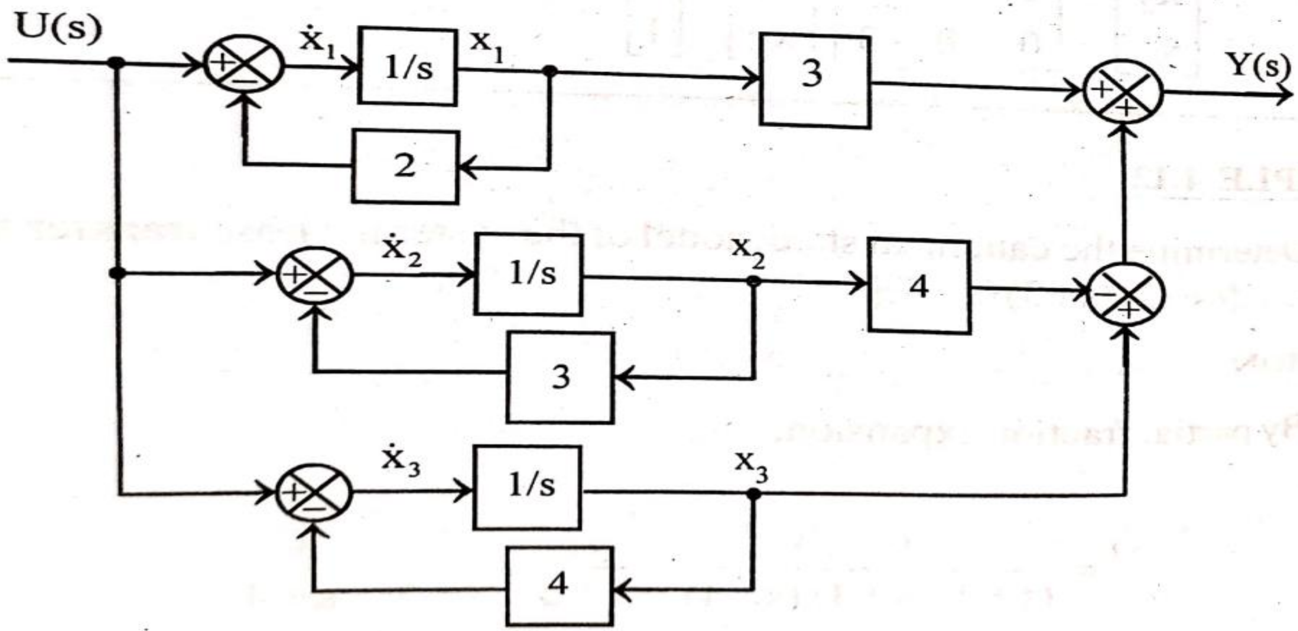
$$= [\frac{\frac{1}{s}}{(1+\frac{1}{s}*2)} \text{ x } 3]U(s) - [\frac{\frac{1}{s}}{(1+\frac{1}{s}*3)} \text{ x } 4]\, U(s) + [\frac{\frac{1}{s}}{(1+\frac{1}{s}*4)}]u(s)$$

The equation can be represented by the block diagram as shown

Assign state variables at the output of the integrators as shown.

At the input of the integrators , first derivative of the state variables are present.

The state equations are formed by adding all the incoming signals to the integrator and equating to the corresponding first derivative of state variables.



The state equations are

$$\dot{x}_1 = -2x_1 + u$$

$$\dot{x}_2 = -3x_2 + u$$

$$\dot{x}_3 = -4x_3 + u$$

The output equation is

$$y = 3x_1 - 4x_2 + x_3$$

The State model is given by,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Matlab

Find the state model for the transfer function $\frac{2(s+5)}{(s+2)(s+3)(s+4)}$ using Matlab

```
>> num=[2 10];
>> den=[1 9 26 24];
>> [A,B,C,D]=tf2ss(num,den)

A =

    -9   -26   -24
     1     0     0
     0     1     0


B =

     1
     0
     0


C =

     0     2    10


D =

     0
```

Solution of State Equation

S-Domain:

The State equation of n^{th} order system is given by,

$\dot{x}(t) = A\,X(t) + BU(t)$; $X(0) = X_0$ = initial condition vector

Taking Laplace transforms on both sides,

$S\,X(s) - X(0) = A\,X(s) + B\,U(s)$

$X(s)[sI-A] = X(0) + B\,U(s)$ where I = unit matrix

$X(s) = [sI-A]^{-1} X(0) + [SI-A]^{-1} B\,U(s)$ --------------(1)

Taking inverse Laplace transforms on both sides,

$X(t) = L^{-1}\,[sI-A]^{-1} X(0) + L^{-1}\,[SI-A]^{-1} B\,U(s)$

where $L^{-1}\,[sI-A]^{-1} = \emptyset(t)$ = state transition matrix

$[sI-A]^{-1} = \emptyset(s)$ = Resolvent matrix

The solution of state equation is,

$X(t) = \emptyset(t)\,X(0) + L^{-1}\,[\emptyset(s).\,B\,U(s)]$

The output equation is,

$$y(t) = C\,x(t) + D\,u(t)$$

Taking Laplace transforms on both sides,

$$Y(s) = C\,X(s) + D\,U(s)$$

From equation (1), $X(s) = [sI\text{-}A]^{-1} X(0) + [SI\text{-}A]^{-1} B\,U(s)$

$$= C\{[sI\text{-}A]^{-1} X(0) + [SI\text{-}A]^{-1} B\,U(s)\} + D\,U(s)$$

$$= C[sI\text{-}A]^{-1} X(0) + C\,[SI\text{-}A]^{-1} B\,U(s) + D\,U(s)$$

For zero initial conditions, $X(0) = 0$

$$\therefore Y(s) = C\,[SI\text{-}A]^{-1} B\,U(s) + D\,U(s)$$

$$= \{C\,[SI\text{-}A]^{-1} B + D\}\,U(s)$$

The transfer function $= \dfrac{Y(s)}{U(s)} = C\,[SI\text{-}A]^{-1} B + D$

Problem

A state variable description of a system is given by the matrix equation,

$$\dot{X} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$Y = [\,1 \quad 1\,]\, X$$

Find (i) The Transfer function

(ii) The State transition matrix

(iii) State diagram

Solution

The state model is given by

$$\dot{X} = A\,X + B\,U$$

$$Y = C\,X + D\,U$$

From the given problem,

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = [\,1 \quad 1\,]$$

(i) The transfer function = $\frac{Y(s)}{U(s)} = C\,[SI-A]^{-1}\,B + D$

Here D =0

$$\therefore \frac{Y(s)}{U(s)} = C\,[SI-A]^{-1}\,B$$

$$[sI\text{-}A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} s+1 & 0 \\ -1 & s+2 \end{bmatrix}$$

$$[SI\text{-}A]^{-1} = \frac{\text{Adj } A}{\text{Det } A} = \begin{bmatrix} s+2 & 0 \\ 1 & s+1 \end{bmatrix} \frac{1}{(s+1)(s+2)}$$

$$\frac{Y(s)}{U(s)} = C\,[SI\text{-}A]^{-1}\,B$$

$$= [1 \quad 1] \begin{bmatrix} s+2 & 0 \\ 1 & s+1 \end{bmatrix} \left\{\frac{1}{(s+1)(s+2)}\right\} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \left\{\frac{1}{(s+1)(s+2)}\right\} [1 \quad 1] \begin{bmatrix} s+2 & 0 \\ 1 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \left\{\frac{1}{(s+1)(s+2)}\right\} [1 \quad 1] \begin{bmatrix} s+2 \\ 1 \end{bmatrix}$$

$$= \left\{\frac{1}{(s+1)(s+2)}\right\} [s+3]$$

$$= \frac{s+3}{(s+1)(s+2)}$$

(ii) State transition matrix = $\emptyset(t) = L^{-1} [sI-A]^{-1}$

$$[SI-A]^{-1} = \begin{bmatrix} s+2 & 0 \\ 1 & s+1 \end{bmatrix} \frac{1}{(s+1)(s+2)}$$

$$= \begin{bmatrix} \frac{s+2}{(s+1)(s+2)} & \frac{0}{(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} & \frac{s+1}{(s+1)(s+2)} \end{bmatrix}$$

$$[SI-A]^{-1} = \begin{bmatrix} \frac{1}{(s+1)} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{(s+2)} \end{bmatrix}$$

$$\emptyset(t) = L^{-1}\,[sI-A]^{-1}$$

$$= L^{-1} \begin{bmatrix} \frac{1}{(s+1)} & 0 \\ \frac{1}{(s+1)(s+2)} & \frac{1}{(s+2)} \end{bmatrix}$$

$$= L^{-1} \begin{bmatrix} \frac{1}{(s+1)} & 0 \\ \frac{1}{(s+1)} - \frac{1}{(s+2)} & \frac{1}{(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-2t} & e^{-2t} \end{bmatrix}$$

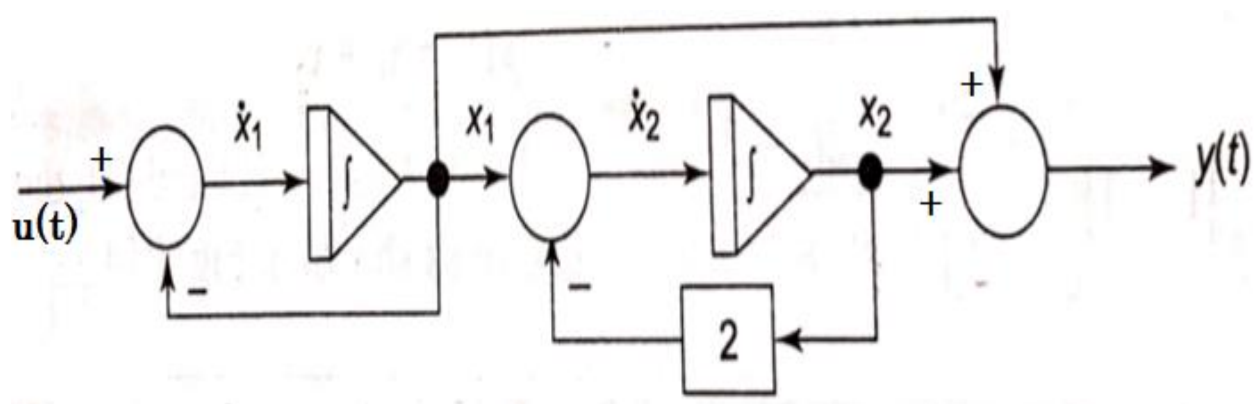
(iii) State Diagram:

The state equation is,

$$\dot{x}_1 = - x_1 + u(t)$$

$$\dot{x}_2 = x_1 - 2x_2$$

The output equation is $y = x_1 + x_2$



Matlab

Find the transfer function using Matlab

```
>> A=[-1,0; 1, -2];
>> B=[1; 0];
>> C=[1 1];
>> D=[0];
>> [num,den]= ss2tf(A,B,C,D)

num =

     0     1     3


den =

     1     3     2

>> g=tf(num,den)

g =

      s + 3
  -------------
  s^2 + 3 s + 2

Continuous-time transfer function.
```

Problem

The state equation of a LTI system is given as

$$\dot{x} = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix} X + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \text{ and } y = [1 \quad 1]\, X$$

Determine (i) State transition matrix

(ii) The transfer function

(iii) State diagram

Solution

From the given system,

$$A = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

(i) The State transition matrix ,

$$\emptyset(t) = L^{-1}\,[sI-A]^{-1}$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} s & -5 \\ 1 & s+2 \end{bmatrix}$$

$$[sI-A]^{-1} = \begin{bmatrix} s+2 & 5 \\ -1 & s \end{bmatrix} \frac{1}{s(s+2)+5} = \begin{bmatrix} \frac{s+2}{s(s+2)+5} & \frac{5}{s(s+2)+5} \\ \frac{-1}{s(s+2)+5} & \frac{s}{s(s+2)+5} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+1+1}{(s+1)^2+2^2} & \dfrac{5}{(s+1)^2+22} \\ \dfrac{-1}{(s+1)^2+22} & \dfrac{s}{(s+1)^2+22} \end{bmatrix}$$

$\emptyset(t) = L^{-1}$ [sI−A]−1

$$= L^{-1} \begin{bmatrix} \dfrac{s+1+1}{(s+1)^2+22} & \dfrac{5}{(s+1)^2+22} \\ \dfrac{-1}{(s+1)^2+22} & \dfrac{s+1-1}{(s+1)^2+22} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}\cos 2t + \frac{1}{2}\, e^{-t}\sin 2t & \frac{5}{2}\, e^{-t}\sin 2t \\ -\frac{1}{2}\, e^{-t}\sin 2t & e^{-t}\cos 2t - \frac{1}{2}\, e^{-t}\sin 2t \end{bmatrix}$$

(ii) The transfer function

$$\frac{Y(s)}{U(s)} = C\,[SI\text{-}A]^{-1}\,B$$

$$= [1 \quad 1] \begin{bmatrix} s+2 & 5 \\ -1 & s \end{bmatrix} \frac{1}{s(s+2)+5} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s(s+2)+5} [1 \quad 1] \begin{bmatrix} S+7 \\ S-1 \end{bmatrix}$$

$$= \frac{2S+6}{s(s+2)+5}$$

Matlab

Find the transfer function using Matlab

```
>> A=[0 5;-1 -2]

A =

     0     5
    -1    -2

>> B=[1;1]

B =

     1
     1

>> C=[1 1]

C =

     1     1

>> D=[0]

D =

     0
```

```
>> [num,den]=ss2tf(A,B,C,D)

num =

     0     2     6


den =

    1.0000    2.0000    5.0000

>> g=tf(num,den)

g =

     2 s + 6
  -------------
  s^2 + 2 s + 5

Continuous-time transfer function.
```

Problem

Find the transfer function of a state model of a system given by ,

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} X$

Solution

From the given system,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[sI-A] = s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$\therefore [sI\text{-}A]^{-1} = \begin{bmatrix} (s+2)(s+1) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix} \frac{1}{s^3+3s^2+2s+1}$$

The transfer function of the system is given by

$$\frac{Y(s)}{U(s)} = C\,[SI\text{-}A]^{-1}\,B$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (s+2)(s+1) & s+3 & 1 \\ -1 & s(s+3) & s \\ -s & -(2s+1) & s^2 \end{bmatrix} \frac{1}{s^3+3s^2+2s+1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s+3 & 1 \\ s(s+3) & s \\ -(2s+1) & s^2 \end{bmatrix} \frac{1}{s^3+3s^2+2s+1}$$

$$= \begin{bmatrix} s+3 & 1 \\ -(2s+1) & s^2 \end{bmatrix} \frac{1}{s^3+3s^2+2s+1}$$

Matlab

Find the transfer function using Matlab

```
>> A=[0 1 0;0 0 1;-1 -2 -3];
>> B=[0 0;0 1;1 0];
>> C=[1 0 0;0 0 1];
>> D=[0 0;0 0];
>> [num,den]=ss2tf(A,B,C,D,1)

num =

     0     0     0     1
     0     1     0     0


den =

    1.0000    3.0000    2.0000    1.0000
```

Problem

A linear time-invariant system is characterized by state equation $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute the solution of the state equation , assuming the initial vector $X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution

From the given system, A = $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

The solution of state equation is,

X(t) = L^{-1} $[sI\text{-}A]^{-1}$ X(0) + L^{-1} $[SI\text{-}A]^{-1}$ B U(s)

Here U= 0

∴ X(t) = L^{-1} $[sI\text{-}A]^{-1}$ X(0)

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$[sI\text{-}A]^{-1} = \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix} \frac{1}{(s-1)^2}$$

$$[sI-A]^{-1} = \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix} \frac{1}{(s-1)^2}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$X(t) = L^{-1}\,[sI-A]^{-1}\,X(0)$$

$$= L^{-1} \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$$

Solution of state equation (Time Domain)

$\dot{x}(t) = A\,x(t) + B\,u(t)$ $\quad\quad\quad\quad$ $X(0) = x_0$

$\dot{x}(t) - A\,x(t) = B\,u(t)$

pre-multiplying both sides by e^{-At}

$e^{-At}[\dot{x}(t) - A\,x(t)] = e^{-At}\,B\,u(t)$ ----------------(1)

Consider, $\frac{d}{dt}\{e^{-At}\,x(t)\} = e^{-At}\,\dot{x}(t) - A\,e^{-At}\,x(t)]$

$= e^{-At}[\dot{x}(t) - A\,x(t)]$

$\therefore$ equation (1) $= \frac{d}{dt}\{e^{-At}\,x(t)\} = e^{-At}\,B\,u(t)$

Integrating with respect to t

$$\int_0^t \frac{d}{dt}\{e^{-At}\,x(t)\}\,dt = \int_0^t [e^{-A\tau}\,B\,u(\tau)]d\tau$$

$$e^{-At}\,x(t) - x(0) = \int_0^t [e^{-A\tau}\,B\,u(\tau)]d\tau$$

Pre-multiplying both sides by e^{At},

$$e^{At}\,[e^{-At}\,x(t) - x(0)] = e^{At}[\int_0^t e^{-A\tau}\,B\,u(\tau)d\tau]$$

$$x(t) = e^{At}[x(0) + \int_0^t e^{-A\tau}\,B\,u(\tau)d\tau]$$

$$= e^{At}\,x(0) + \int_0^t [e^{A(t-\tau)}\,B\,u(\tau)]d\tau$$

$$x(t) = \emptyset(t)\,x(0) + \int_0^t \emptyset(t-\tau)\,B\,u(\tau)\,d\tau$$

if the initial time is $t = t_0$, the solution of state equation becomes,

$$x(t) = \emptyset(t - t_0)\,x(t_0) + \int_0^t \emptyset(t-\tau)\,B\,u(\tau)d\tau$$

Properties of state transition matrix

$$\emptyset(t) = e^{At} = L^{-1}\,[sI-A]^{-1}$$

1. $\emptyset(0) = I$
2. $\emptyset^{-1}(t) = \emptyset(-t)$
3. $\emptyset(t_2-t_1)\,\emptyset(t_1-t_0) = \emptyset(t_2-t_0)$ for any t_2, t_1, t_0
4. $[\emptyset(t)]^k = \emptyset(kt)$
5. $\emptyset(t_1+t_2) = \emptyset(t_1)\,\emptyset(t_2) = \emptyset(t_2)\,\emptyset(t_1)$

Problem

Compute the State transition matrix by infinite series method $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

Solution: For the given system matrix A, the state transition matrix is,

$$\emptyset\ (t) = e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \text{------}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$A^2 = A.\ A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$$

$$\emptyset(t) = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} t + \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} \frac{t^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1 - \frac{t^2}{2} + \frac{t^3}{3} + \cdots & t - t^2 + \frac{t^3}{2} + \cdots \\ -t + t^2 - \frac{t^3}{2} + \cdots & 1 - 2t + \frac{3t^2}{2} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} + te^{-t} & te^{-t} \\ -te^{-t} & e^{-t} - te^{-t} \end{bmatrix}$$

Problem

Find the state transition matrix by infinite series method for the system matrix A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Solution: For the given system matrix A, the state transition matrix is,

$$\emptyset\,(t) = e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \text{------}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 . A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\emptyset(t) = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + ---$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} t + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \frac{t^3}{3!} + ---$$

$$= \begin{bmatrix} 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots & t + t^2 + \frac{t^3}{2} + \cdots \\ 0 & 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + .. \end{bmatrix}$$

$$= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$$

References

1. Control Engineering by Nagrath & Gopal, New Age International Publishers
2. Engineering control systems - Norman S. Nise, John WILEY & sons , fifth Edition
3. Modern control Engineering-Ogata, Prentice Hall
4. Automatic Control Systems- B.C Kuo, John Wiley and Sons