B.M.S. COLLEGE OF ENGINEERING
DEPARTMENT OF MATHEMATICS

For the Course Code: 23MA1BSMCS, 23MA1BSCEM

Course: Mathematical Foundation for Computer Science Stream – 1
Mathematical Foundation for Civil, Electrical and Mechanical Engg. Stream - 1

I. RANK OF MATRICES

1. Find the rank of the following matrices by reducing them into echelon form:

a) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$.
Ans: ρ=2

b) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$.
Ans: ρ=2

c) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.
Ans: ρ=3

d) $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$.
Ans: ρ=3

e) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$.
Ans: ρ=2

f) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$.
Ans: ρ=2

g) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$.
Ans: ρ=2

h) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$.
Ans: ρ=3

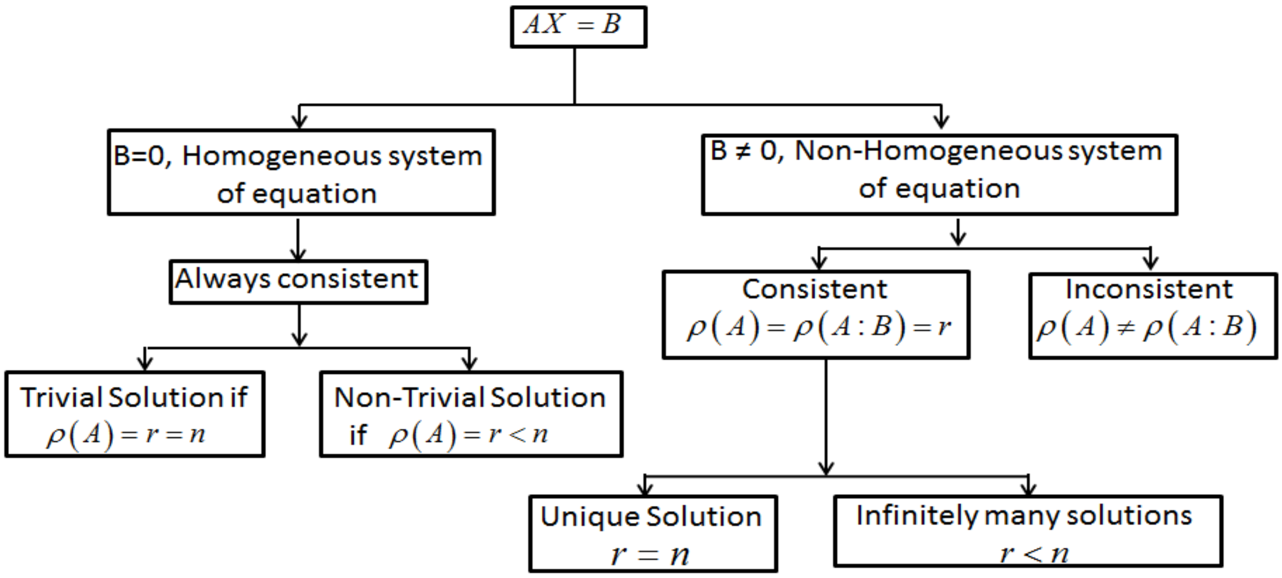
i) $\begin{bmatrix} 1 & -2 \\ 3 & -6 \\ 7 & -1 \\ 4 & 5 \end{bmatrix}$

j) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$.
Ans: ρ=2

k) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$.
Ans: ρ=4

2. Find 'b' if the rank of $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$ is 3. Ans: $b = 2$ or $b = -6$.

II. Consistency and solution of linear system of equations:



where n is number of unknowns.

1. Discuss the consistency of the following system of linear equations

a) $x+2y+3z=0$, $3x+4y+4z=0$, $7x+10y+12z=0$

Ans: Consistent and has trivial solution.

b) $2x_1+3x_2-4x_3+x_4=0$, $x_1-x_2+x_3+2x_4=0$, $5x_1-x_3+7x_4=0$, $7x_1+8x_2-11x_3+5x_4=0$

Ans: Consistent and has infinitely many solutions

$$x_1=\frac{(k_1-7k_2)}{5}, x_2=\frac{(6k_1+3k_2)}{5}, x_3=k_1, x_4=k_2$$

c) $5x+3y+7z=4$, $3x+26y+2z=9$, $7x+2y+10z=5$

Ans: Consistent and has infinitely many solutions

$$x=\frac{7-16k}{11} \quad y=\frac{k+3}{11} \quad z=k$$

d) $4x-2y+6z=8$, $x+y-3z=-1$, $15x-3y+9z=21$

Ans: Consistent and has infinitely many solutions

$x=1,\ y=3k-2,\ z=k$

e) $2x-3y+7z=5$, $3x+y-3z=13$, $2x+19y-47z=32$

Ans: inconsistent

f) $2x_1+3x_2-x_3=1$, $3x_1-4x_2+3x_3=-1$, $2x_1-x_2+2x_3=-3$, $3x_1+1x_2-2x_3=4$

Ans: Inconsistent

g) $x+2y+z=3$, $2x+3y+2z=5$, $3x-5y+5z=2$, $3x+9y-z=4$

Ans: Consistent and has unique solution

$x=-1,\ y=1,\ z=2$

h) $2x+6y+11=0$, $6x+20y-6z+3=0$, $6y-18z+1=0$

Ans: inconsistent

i) $$\begin{aligned}2x_1-2x_2+4x_3+3x_4&=9\\ x_1-x_2+2x_3+2x_4&=6\\ 2x_1-2x_2+x_3+2x_4&=3\\ x_1-x_2+x_4&=2\end{aligned}$$
Ans: Inconsistent

j) $$\begin{aligned}2x+y-z&=0\\ 2x+5y+7z&=52\\ x+y+z&=9\end{aligned}$$
Ans: Consistent and has unique solution
$x=1, y=3, z=5$

k) $$\begin{aligned}3x+2y+2z&=0\\ x+2y&=4\\ 10y+3z&=-2\\ 2x-3y-z&=5\end{aligned}$$
Ans: Consistent and has unique solution
$x=2, y=1, z=-4$

l) $$\begin{aligned}x_1+x_2-2x_3+x_4+3x_5&=1\\ 2x_1-x_2+2x_3+2x_4+6x_5&=2\\ 3x_1+2x_2-4x_3-3x_4-9x_5&=3\end{aligned}$$
Ans: Consistent and has infinitely many solutions
$x_1=1,\ x_2=2a,\ x_3=a,\ x_4=-3b,\ x_5=b$

2. Find the values of λ and μ so that the equations $$\begin{aligned}2x+3y+5z&=9\\ 7x+3y-2z&=8\\ 2x+3y+\lambda z&=\mu\end{aligned}$$ have

(i) no solution (ii) unique solution (iii) infinite number of solutions.

Ans: (i) If $\lambda=5$ and $\mu\neq 9$ (ii) $\lambda\neq 5$ and μ can be any value (iii) $\lambda=5$ and $\mu=9$.

3. Determine the values of a and b for which the system $$\begin{aligned}x+2y+3z&=6\\ x+3y+5z&=9\\ 2x+5y+az&=b\end{aligned}$$ have

(i) no solution (ii) unique solution (iii) infinite number of solutions.

Ans: (i) If $a=8$ and $b\neq 15$ (ii) $a\neq 8$ and for any b (iii) $a=8$ and $b=15$.

4. Find the value of a for which the system $x+2y+z=3;\ ay+5z=10;\ 2x+7y+az=b$ has a unique solution. Also find the pair of values (a,b) for which it has infinitely many solutions.

Ans:

5. Find the values of a and b for which the system of equations $$\begin{aligned}x+ay+z&=3\\ x+2y+2z&=b\\ x+5y+3z&=9\end{aligned}$$ is consistent.

Ans: If $a=-1$ and b =6 equations will be consistent and have infinite number of solutions.

If $a\neq -1$ and b has any value, equations will be consistent and have a unique solution.

6. Show that the equations $$\begin{aligned}3x+4y+5z&=a\\ 4x+5y+6z&=b\\ 5x+6y+7z&=c\end{aligned}$$ do not have a solution unless $a+c=2b$.

7. Test for consistency $$\begin{aligned}-2x+y+z&=a\\ x-2y+z&=b\\ x+y-2z&=c\end{aligned}$$ where a,b and c are constants.

Ans: if $a+b+c\neq 0$ inconsistent, if $a+b+c=0$, then infinitely many solution.

III. Gauss elimination method:

1. Solve the following system of equations by Gauss elimination method:

a)
$$2x_1 + x_2 + x_3 = 10$$
$$3x_1 + 2x_2 + 3x_3 = 18$$
$$x_1 + 4x_2 + 9x_3 = 16$$
Ans: $x_1 = 7,\ x_2 = -9,\ x_3 = 5.$

b)
$$2x + 2y + z = 12$$
$$3x + 2y + 2z = 8$$
$$5x + 10y - 8z = 10$$
Ans: $x = \frac{-51}{4}, y = \frac{115}{8}, z = \frac{35}{4}$

c)
$$2x_1 + 4x_2 + x_3 = 3$$
$$3x_1 + 2x_2 - 2x_3 = -2$$
$$x_1 - x_2 + x_3 = 6$$
Ans: $x_1 = 2,\ x_2 = -1,\ x_3 = 3.$

d)
$$10x + 2y + z = 9$$
$$2x + 20y - 2z = -44$$
$$-2x + 3y + 10z = 22$$
Ans: $x = 1, y = -2, z = 3$

e)
$$2x_1 + x_2 + 4x_3 = 12$$
$$8x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$
Ans: $x_1 = 3,\ x_2 = 2,\ x_3 = 1.$

f)
$$x_1 + 4x_2 - x_3 = -5$$
$$x_1 + x_2 - 6x_3 = -12$$
$$3x_1 - x_2 - x_3 = 4$$
Ans: $x_1 = \frac{117}{71},\ x_2 = -\frac{81}{71},\ x_3 = \frac{148}{71}.$

g)
$$2x_1 + x_2 + 3x_3 = 1$$
$$4x_1 + 4x_2 + 7x_3 = 1$$
$$2x_1 + 5x_2 + 9x_3 = 3$$
Ans: $x_1 = -1/2,\ x_2 = -1,\ x_3 = 1.$

h)
$$2x_1 - 7x_2 + 4x_3 = 9$$
$$x_1 + 9x_2 - 6x_3 = 1$$
$$-3x_1 + 8x_2 + 5x_3 = 6$$
Ans: $x_1 = 4, x_2 = 1, x_3 = 2$

i)
$$5x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 7x_2 + x_3 + x_4 = 12$$
$$x_1 + x_2 + 6x_3 + x_4 = -5$$
$$x_1 + x_2 + x_3 + 4x_4 = -6$$
Ans: $x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = -2$

2. Show that if $\lambda \neq -5$ the system of equations
$$3x - y + 4z = 3$$
$$x + 2y - 3z = -2$$
$$6x + 5y + \lambda z = -3$$
have a unique solution. If $\lambda = -5$ show that the equations are consistent. Determine the solution in each case.

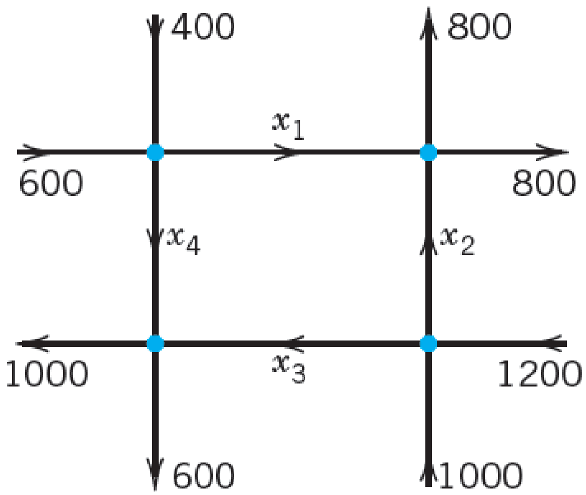
Ans: when $\lambda \neq -5,\ x = \frac{4}{7},\ y = \frac{-9}{7},\ z = 0$, when $\lambda = -5,\ x = \frac{4-5k}{7},\ y = \frac{13k-9}{7},\ z = k$.

3. Prove that the equations
$$5x + 3y + 2z = 12$$
$$2x + 4y + 5z = 2$$
$$39x + 43y + 45z = c$$
are incompatible unless $c = 74$; and in that case the equations are satisfied by $x = 2 + t, y = 2 - 3t, z = -2 + 2t$, where t is any arbitrary quantity.

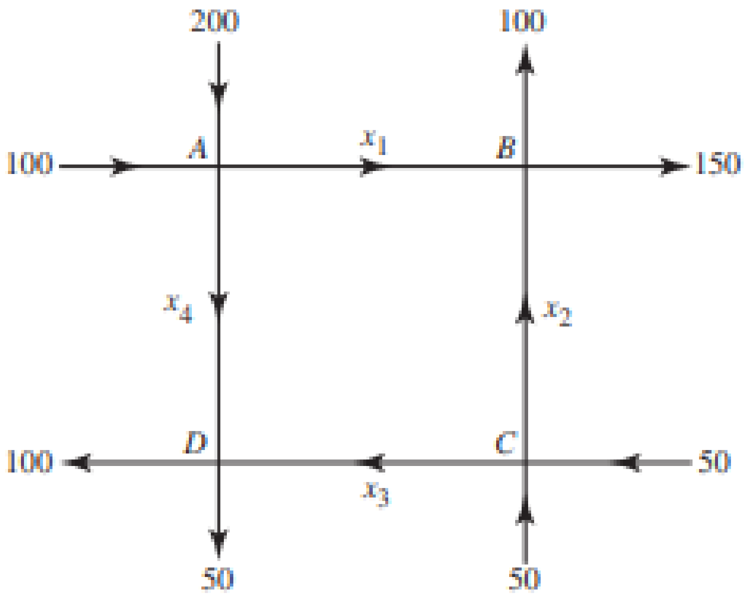
4. For what values of k the equations $\begin{aligned} x+y+z&=1 \\ 2x+y+4z&=k \\ 4x+y+10z&=k^2 \end{aligned}$ have a solution and solve them completely in each case. **Ans**: when $k=1$, $x=-3z, y=2z+1$, when $k=2$, $x=1-3z, y=2z.$

APPLICATIONS

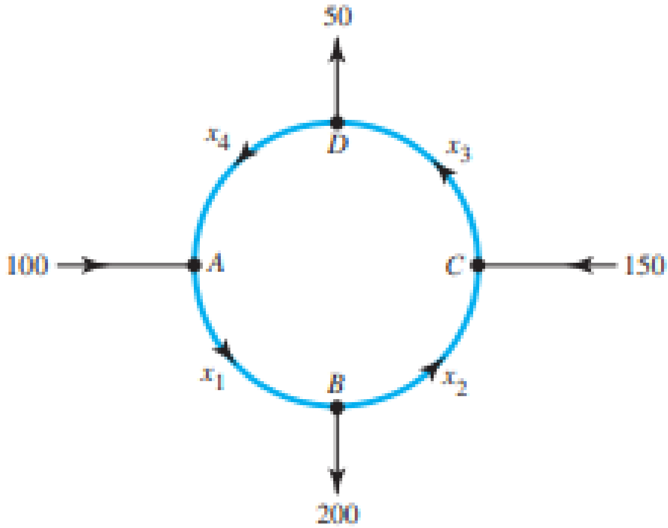
1. Find the traffic flow in the net of one-way streets directions shown in the figure



2. Find the traffic flow in the net of one-way street directions shown in figure



3. Following figure represents the traffic entering and leaving a roundabout road junction. Such junctions are common in Basavanagudi. Construct a mathematical model that describes the flow of traffic along the various branches. What is minimum flow theoretically possible along branch BC? What are the other flows at that time? (The units of flow are vehicles per hour).



4. Balance the chemical equations by solving corresponding system of equations
   a) $N_2 + O_2 \to N_2O_5$
   b) $H_2O \to H_2 + O_2$
   c) $C_2H_6 + O_2 \to H_2O + CO_2$

IV. Gauss-Seidel Iteration method

The solution converges if the system is diagonally dominant. Suppose $AX = B$ is diagonally dominant with $A = (a_{ij})$, $X = (x_i)$ and $B = (b_i)$. Then the iterative formula is

$$x_1^{n+1} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^n - a_{13}x_3^n \cdots\cdots\right)$$

$$x_2^{n+1} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{n+1} - a_{23}x_3^n \cdots\cdots\right)$$

$$x_3^{n+1} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{n+1} - a_{32}x_2^{n+1} \cdots\cdots\right)$$

$$\vdots$$

Solve the system of equations using Gauss-Seidel method:

a.
$20x + y - 2z = 17$
$3x + 20y - z = -18$
$2x - 3y + 20z = 25$
Ans: $x = 1,\ y = -1,\ z = 1$

b.
$5x + 2y + z = 12$
$x + 4y + 2z = 15$
$x + 2y + 5z = 20$
Ans: $x = 0.996,\ y = 2,\ z = 3$

c.
$2x + y + 6z = 9$
$8x + 3y + 2z = 13$
$x + 5y + z = 7$
Ans: $x = 1,\ y = 1,\ z = 1$

d.
$28x + 4y - z = 32$
$x + 3y + 10z = 24$
$2x + 17y + 4z = 35$
Ans: $x = 0.9876,\ y = 1.5090,\ z = 1.8485$

e. $10x+2y+z=9$
$2x+20y-2z=-44$
$-2x+3y+10z=22$
Ans: $x=1,\ y=-2,\ z=3$

f. $83x+11y-4z=95$
$7x+52y+13z=104$
$3x+8y+29z=71$
Ans: $x=1.06, y=1.37, z=1.96$

g. $54x+y+z=110$
$2x+15y+6z=72$
$-x+6y+27z=85$
Ans: $x=1.926,\ y=3.573,\ z=2.425$

h. $5x_1-x_2=9$
$-x_1+5x_2-x_3=4$
$-x_2+5x_3=-6$
Ans: $x_1=1.99,\ x_2=0.99,\ x_3=-1$

i. $8x_1+x_2-x_3=8$
$2x_1+x_2+9x_3=12$
$x_1-7x_2+2x_3=-4$
Ans: $x_1=1\ x_2=1,\ x_3=1$

j. $4x_1+2x_2+x_3=11$
$-x_1+2x_2=3$
$2x_1+x_2+4x_3=16$
Ans: $x_1=1,\ x_2=2, x_3=3$

k. Start with $(2,2,-1)$ and solve
$5x_1-x_2+x_3=10$
$2x_1+4x_2=12$
$x_1+x_2+5x_3=-1$
$Ans: x_1=2.5555, x_2=1.7222,\ x_3=-1.0555$

l. $10x_1+x_2+x_3=12$
$2x_1+10x_2+x_3=13$
$2x_1+2x_2+10x_3=14$
Ans: $x_1=x_2=x_3=1$

m. $27x_1+6x_2-x_3=85$
$6x_1+15x_2+2x_3=72$
$x_1+x_2+54x_3=110$
Ans: $x_1=2.4255,\ x_2=3.573,\ x_3=1.926$

n. $x_1-8x_2+3x_3=-4$
$2x_1+x_2+9x_3=12$
$8x_1+2x_2-2x_3=8$
$Ans: x_1=x_2=x_3=1$

o. $$\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\\u\end{bmatrix}=\begin{bmatrix}0\\0\\0\\5\end{bmatrix}$$
Ans: $x=1,\ y=2,\ z=3, u=4$

V. Characteristic values (Eigen values) and characteristic vectors (Eigen vectors)

If A is a square matrix, then λ is said to be an eigen value of the matrix if there exists a non-zero vector X such that $AX=\lambda X$. X is called the Eigen vector corresponding to the Eigen value λ .

$\lambda X=\lambda IX \Rightarrow (A-\lambda I)X=0$. We seek non-trivial solution of $(A-\lambda I)X=0$.

X is non-trivial if $\rho(A-\lambda I)<n \Rightarrow |A-\lambda I|=0$.

If A is matrix of size 3×3 then $-\lambda^3+Tr(A)\lambda^2-\sum M_{ii}^{2\times 2}\lambda+|A|=0$.

Find the Eigen values and Eigen vectors of the following matrices:

a. $$\begin{bmatrix}1&1&3\\1&5&1\\3&1&1\end{bmatrix}$$
Ans: $\lambda=-2,3,6; x_1=[-k,0,k],$.
$x_2=[k,-k,k],\ x_3=[k,2k,k]$

b. $$\begin{bmatrix}3&1&4\\0&2&6\\0&0&5\end{bmatrix}$$
Ans: $\lambda=2,3,5\ \ x_1=k_1[1,-1,0]$
$x_2=k_2[1,0,0],\ x_3=k_3[3,2,1]$

c. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Ans: $\lambda = 0, 3, 15 \quad x_1 = [1, 2, 2],$
$x_2 = [2, 1, -2], x_3 = [2, -2, 1]$

d. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Ans: $\lambda = 1, 2, 3 \quad x_1 = [1, 0, -1]$
$x_2 = [0, 1, 0], x_3 = [1, 0, 1]$

e. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Ans: $\lambda = 5, -3, -3 \quad x_1 = k[1, 2, -1]$.
$x_2 = [3k_1 - 2k_2, k_2, k_1]$

f. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Ans: $\lambda = 8, 2, 2;\ x_1 = [2, -1, 1],$.
$x_2 = [1, 0, -2], x_3 = [1, 2, 0]$

g. $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

Ans: $\lambda = 1, -1, 4;\ x_1 = [2, -1, 1],$.
$x_2 = [0, 1, 1], x_3 = [-1, -1, 1]$

h. $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

Ans: $\lambda = 1, 2, 2;\ x_1 = k_1[1, 1, -1],$.
$x_2 = k_2[2, 1, 0], x_3 = k_3[2, 1, 0]$

i. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Ans: $\lambda = 5, 1, 1;\ x_1 = [1, 1, 1],$.
$x_2 = [1, 0, -1], x_3 = [2, -1, 0]$

j. $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$

Ans: $\lambda = 2, 2, 3$
$x_1 = x_2 = [5, 2, -3], x_3 = [1, 1, -2]$

k. $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Ans: $\lambda = 2, 2, -2$
$x_1 = x_2 = [0, 1, 1] \quad x_3 = [-4, -1, 1]$

l. $\begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix}$

Ans: $\lambda = -5, 2, 2$
$x_1 = [3, 2, 4],\ x_2 = x_3 = [1, 3, -1]$

VI. Rayleigh Power Method to find Dominant Eigenvalue and eigenvector

Find AX_i and express $AX_i = |\lambda_i| X_{i+1}$ where $|\lambda_i|$ is the numerically largest element of X_i. Iterate until desired accuracy.

Largest/dominant Eigen value is given by $\lambda = \dfrac{\langle AX_n, X_n \rangle}{\langle X_n, X_n \rangle}$.

I. Find the dominant Eigen value and the corresponding Eigen vector of following matrices.

1. $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 9 \\ 0 & 1 & 9 \end{bmatrix}$ with the initial vector $[1 \ 0 \ 0]^T$
2. A= $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ by taking initial vector as $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$
3. A= $\begin{bmatrix} -0.5 & 0.5 & -1 \\ 0.5 & -0.5 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ by taking initial vector as $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$. Ans: $\lambda \approx\sim -2$. Others are 1 and 0
4. $A = \begin{bmatrix} 8 & 0 & 12 \\ 1 & -2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ with the initial vector $[1 \ 0 \ 0]^T$