B.M.S. COLLEGE OF ENGINEERING, BENGALURU
DEPARTMENT OF MATHEMATICS

For the Course Code: 22MA1BSMCV, 22MA1BSMES & 22MA1BSMME

ORDINARY DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Linear differential equations are those in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

The general n[th] order linear differential equation is of the form

$$p_0 \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + p_n y = X \qquad \ldots\ldots\ldots(1)$$

where $p_0, p_1, p_2, \ldots\ldots p_n$ and X are all functions of x only.

In the above differential equation if

i) $X = 0$ Then it is called a homogeneous linear differential equation.
ii) $X \neq 0$ Then it is called a non-homogeneous linear differential equation.
iii) $p_0, p_1, p_2, \ldots\ldots p_n$ are constants then it is called a linear ordinary differential equation with constant coefficients.
iv) If $y_0, y_1, \ldots\ldots, y_n$ are n independent solutions then its linear combination $y = c_1 y_1, + c_2 y_2 + \cdots + c_n y_n$ is also a solution.

Linear ordinary differential equations with constant coefficients

$$p_0 \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} p_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots \; p_n y = X$$

Where $p_0, p_1, p_2, \ldots\ldots p_n$ are all constants.

Note:

I) Solution: A relation between *x* and *y* which satisfies the given differential equation.

II) General solution: A solution which contains arbitrary constants.

iii) $p_0, p_1, p_2, \ldots\ldots p_n$ are functions of x then it is called a linear ordinary differential equation with variable coefficients.

iv) If $y = v$ be any particular solution and $y = u$ be the solution of the homogeneous part of the differential equation then $y = u + v$ is called as the **complete solution**.

The part u is called the **complementary function** (**C.F.**) and the part v is called the **particular integral** (**P.I.**). Thus the complete solution of the above differential equation is $y = \text{C.F} + \text{P.I}$. **Particular integral (P.I.)** is a function which satisfies the ODE given by equation **(1).**

RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Let us consider n[th] order homogeneous linear differential equation with constant coefficients

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + k_n y = 0 \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

The equation (2) in symbolic form is $(D^n + k_1D^{n-1} + k_2D^{n-2} ... + k_n)y = 0$ ..………….. (3)

Further which can be written as $f(D)y = X$

where$f(D) = D^n + k_1D^{n-1} + k_2D^{n-2} ... + k_n$ …….. (4)

Observation:

Auxiliary equation (A.E.):$F(m) = 0$. (Obtained by assuming e^{mx} as a solution of $f(D)y = 0$.

Therefore the auxiliary equation of (2) is $m^n + k_1m^{n-1} + k_2m^{n-2} ... + k_n = 0$ …………….. (5)

Let $m_1, m_2 ... ... m_n$ be the roots of equation (5).

Case I: A*ll the roots be real & distinct*

The complete solution of (3) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + ... + c_ne^{m_nx}$$

Case II: *The roots are real and repeated* (*equal*)

Suppose if two roots are equal (i.e., $m_1=m_2=m$) then the complete solution of (3) is

$$y = (c_1 + c_2x)e^{mx} + c_3e^{m_3x} + ... + c_ne^{m_nx}.$$

Similarly if r number of roots are equal (i.e., $m_1=m_2=.........=m_r=m$) then the complete solution of (3) is

$$y = (c_1 + c_2x + c_3x^2 + ... + c_rx^{r-1})e^{mx} + c_{r+1}e^{m_{r+1}x} + ... + c_ne^{m_nx}$$

Case III: *The roots are imaginary*

Suppose if one pair of roots be imaginary (*i.e.*, $m_1=\alpha+i\beta$, $m_2=\alpha-i\beta$) then the complete solution of (3) is

$$y = (c_1\cos\beta x + c_2\sin\beta x)e^{\alpha x} + c_3e^{m_3x} + ... + c_ne^{m_nx}.$$

Suppose if two points of imaginary roots are equal (*i.e.*, $m_1 = m_2 = \alpha + i\beta, m_3 = m_4 = \alpha - i\beta$) then the complete solution of (3) is

$$y = \left[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x\right]e^{\alpha x} + c_5e^{m_5x} .... + c_ne^{m_nx}.$$

Solve the following Homogeneous Linear Differential equations with constant coefficients:

1. $y'' - y' - 12y = 0$ **Ans:** $y = c_1e^{-3x} + c_2e^{4x}$
2. $y'' - 4y' + 4y = 0$, $y(0) = 3,\ y'(0) = 1$ **Ans:** $y = (3 - 5x)e^{2x}$
3. $4y''' + 4y'' + y' = 0$ **Ans:** $y = c_1 + (c_2 + c_3x)e^{-x/2}$
4. $\dfrac{d^3y}{dx^3} + y = 0$ **Ans:** $y = c_1e^{-x} + \left(c_2\cos\dfrac{\sqrt{3}x}{2} + c_3\sin\dfrac{\sqrt{3}x}{2}\right)e^{x/2}$
5. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$ **Ans:** $y = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$
6. $4y'' - 12y' + 5y = 0$ **Ans:** $y = c_1e^{x/2} + c_2e^{5x/2}$

7. $y''' - 4y'' + y' + 6y = 0$ **Ans:** $y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{3x}$

8. $y'' + 6y'' + 9y = 0$ **Ans:** $y = (c_1 + c_2 x)e^{-3x}$.

9. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$ **Ans:** $y = (c_1 + c_2 x + c_3 x^2)e^{x}$.

10. $(D^2+1)^3 y = 0$ **Ans:** $y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x$.

Rules for finding the particular integral (P. I.) of

The nth order non-homogeneous linear differential equation with constant coefficients can be expressed as

$$\frac{d^ny}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} k_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots k_n y = X \quad \ldots\ldots(6)$$

The equation (6) in symbolic form is

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n)y = X \quad \ldots\ldots(7)$$

Further which can be written as $f(D)y = X$ where

$$f(D) = D^n + k_1 D^{n-1} + k_2 D^{n-2} + \cdots + k_n$$

Particular integral of is given by $P.I. = \frac{1}{f(D)} X$

Case I: When $X = e^{ax}$

$$\text{P.I.} = \frac{1}{f(D)} X = \frac{1}{f(a)} e^{ax}, \quad \text{provided } f(a) \neq 0$$

If $f(a) = 0$, $\text{P.I.} = x\frac{1}{f'(a)} e^{ax}$, provided $f'(a) \neq 0$

If $f'(a) = 0$, $\text{P.I.} = x^2\frac{1}{f''(a)} e^{ax}$, provided $f''(a) \neq 0$ and so on.

Case II: When $X = \cos(ax+b)$ or $\sin(axb)$

$$\text{P.I.} = \frac{1}{f(-a^2)}\left[\cos(ax+b) \text{ or } \sin(ax+b)\right], \quad \text{provided } f(-a^2) \neq 0.$$

If $f(-a^2) = 0$, $\text{P.I.} = x\frac{1}{f'(-a^2)}\left[\cos(ax+b) \text{ or } \sin(ax+b)\right]$, provided $f'(-a^2) \neq 0$.

If $f'(-a^2) = 0$, $\text{P.I.} = x^2\frac{1}{f''(-a^2)}\left[\cos(ax+b) \text{ or } \sin(ax+b)\right]$, provided $f''(-a^2) \neq 0$

Case III: When $X = x^m$

$$\text{P.I.} = \frac{1}{f(D)} x^m = [f(D)]^{-1} x^m.$$

Expand $[f(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since the $D^n x^m = 0$ for $n > m$, we need not consider terms beyond D^m.

Solve the following non-Homogeneous Linear Differential equations with constant coefficients:

Problems on Case I

1. $y_2 - 6y_1 + 9y = 6e^{3x} + 7e^{-2x} - \log 2$. **Ans:** $y = (c_1 + c_2 x)e^{3x} + 3x^2 e^{3x} + \frac{7}{25}e^{-2x} - \frac{1}{9}\log 2$

2. $(D-2)^3 y = e^{2x}$. **Ans:** $y = \left(c_1 + c_2 x + c_3 x^2\right)e^{2x} + \frac{x^3}{6}e^{2x}$

3. $\left(D^2+4\right) y = e^{3x}$. **Ans:** $y = \left(c_1 e^{2ix} + c_2 e^{-2ix}\right) + \frac{1}{13}e^{3x}$

4. $\left(D^2 - D + 1\right) y = \sinh x$. **Ans:** $y = \left(c_1 \cos\frac{\sqrt{3}x}{2} + c_2 \sin\frac{\sqrt{3}x}{2}\right)e^{x/2} + \frac{1}{6}\left(3e^x - e^{-x}\right)$

5. $D(D+1)^2 y = 12e^{-x}$. **Ans:** $y = c_1 + (c_2 + c_3 x)e^{-x} - 6x^2 e^{-x}$

6. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -\cosh x, \quad y=0, \ \frac{dy}{dx} = 1 \ \text{at} \ x=0$. **Ans:** $y = \frac{3}{5}e^{-2x}(\cos x + 3\sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2}$.

7. $\left(D^2 + 2D + 1\right) y = 2e^{3x}$. **Ans:** $y = (c_1 + c_2 x)e^{-x} + \frac{e^{3x}}{8}$

8. $(D-2)^3 y = e^{2x}$. **Ans:** $y = \left(c_1 + c_2 x + c_3 x^2\right)e^{2x} + \frac{x^3 e^{2x}}{6}$

9. $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$. **Ans:** $y = c_1 e^{-2x} + (c_2 + c_3 x)e^x + \frac{xe^{-2x}}{9} + \frac{x^2 e^x}{6} + \frac{1}{4}e^{-x}$

Problems on Case II

1. $\left(D^2+4\right) y = \sin 3x + \cos 2x$. **Ans:** $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{5}\sin 3x + \frac{x}{4}\sin 2x$

2. $\left(D^2 + 5D - 6\right) y = \sin(4x)\sin x$ **Ans:** $y = c_1 e^x + c_2 e^{-6x} + \frac{1}{2}\left[\frac{\sin 3x - \cos 3x}{30} + \frac{31\cos 5x - 25\sin 5x}{1586}\right]$.

3. $\frac{d^2x}{dt^2} + n^2 x = k\cos(nt + \alpha)$. **Ans:** $x = c_1 \cos nt + c_2 \sin nt + \frac{kt}{2n}\sin(nt + \alpha)$

4. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$. **Ans:** $y = (c_1 + c_2 x)e^{-x} + \frac{e^{2x}}{9} - \frac{1}{2} - \frac{2}{25}\sin 2x + \frac{3}{50}\cos 2x$

5. $\left(D^3 + 4D\right) y = \sin 2x$. **Ans:** $y = c_1 + c_2 \cos 2x + +c_3 \sin 2x - \frac{x}{8}\sin 2x$

6. $y'' + 4y' + 4y = 3\sin x + 4\cos x, y(0) = 1 \text{and } y'(0) = 0$ **Ans:** $y = (1+x)e^{-2x} + \sin x$

7. Find the PI of $\left(D^3+1\right) y = \cos(2x-1)$. **Ans:** $\text{P.I.} = \frac{1}{65}\left[\cos(2x-1) - 8\sin(2x-1)\right]$

8. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$. **Ans:** $y = c_1 + (c_2 + c_3 x)e^{-x} - \frac{x^2 e^{-x}}{2} + \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x$

Problems on Case III

1. $\left(D^2 - 1\right) y = 2x^4 - 3x + 1$ **Ans:** $y = c_1 e^x + c_2 e^{-x} - \left[2x^4 + 24x^2 - 3x + 49\right]$.

2. $\left(D^6 - D^4\right) y = x^2$ **Ans:** $y = \left(c_1 + c_2 x + c_3 x^2 + c_4 x^3\right) + c_5 e^x + c_6 e^{-x} - \left[\frac{x^6}{360} + \frac{x^4}{12} + x^2 + 2\right]$.

3. $\left(D^2 + 4D + 4\right) y = x^2 + 2x, \ y(0) = 0, \ y'(0) = 0$ **Ans:** $y = -\frac{3}{8}(1+2x)e^{-2x} + \frac{1}{8}\left(2x^2 + 3\right)$.

4. $(D-2)^2 y = 8x^2$. **Ans:** $y = (c_1 + c_2 x)e^{2x} + 2x^2 + 4x + 3$.

5. $\left(D^2+2\right)y=x^3+e^{-2x}+\cos 3x$ **Ans:** $y=(c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x)+\frac{1}{2}\left[x^3-3x\right]+\frac{1}{6}e^{-2x}-\frac{1}{7}\cos 3x$.

6. $\left(D^2+D+1\right)y=x^3$. **Ans:** $y=e^{\frac{x}{2}}(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x)+x^3-3x^2+6$

Alternative method for finding Particular Integral - Method of variation of parameters

This method is quite general and applied to equations of the form

$y''+Py'+Qy=R$ where P,Q and R are functions of x.

Let $\text{C.F.}=c_1u(x)+c_2v(x)$

Then $\text{P.I.}=A(x)u(x)+B(x)v(x)$

Where $A(x)=-\int\frac{vR}{W}dx$, $B(x)=\int\frac{uR}{W}dx$ and $W=\begin{vmatrix}u & v\\ u' & v'\end{vmatrix}$ is called the Wronskian of u,v. Hence the complete solution is given by $y=\text{C.F.}+\text{P.I.}$

Problems

1. $y''+y=\sec x$ **Ans:** $y=c_1\cos x+c_2\sin x+\cos x\log(\cos x)+x\sin x$

2. $\frac{d^2y}{dx^2}+y=\frac{1}{1+\sin x}$ **Ans:** $y=c_1\cos x+c_2\sin x+\sin x\log(1+\sin x)-x\cos x-1$

3. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}=e^x\sin x$ **Ans:** $y=c_1+c_2e^{2x}-\frac{1}{2}e^x\sin x$

4. $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=\frac{1}{1+e^{-x}}$ **Ans:** $y=\left(e^x+e^{2x}\right)\log\left(1+e^x\right)+\left(c_1-1-x\right)e^x+\left(c_2-x\right)e^{2x}$.

5. $\left(D^2+2D+1\right)y=e^{-x}\log x$. **Ans:** $y=c_1e^{-x}+c_2xe^{-x}+\frac{x^2}{2}\left(\frac{1}{2}-\log x\right)e^{-x}+e^{-2x}\left(x\log x-x\right)$

6. $\frac{d^2y}{dx^2}-y=\frac{2}{\left(1+e^x\right)}$ **Ans:** $y=c_1e^x+c_2e^{-x}-1+e^x\log(e^{-x}+1)-e^{-x}\log\left(e^x+1\right)$.

7. $y''-6y'+9y=\frac{e^{3x}}{x^2}$ **Ans:** $y=\left(c_1+c_2x\right)e^{3x}-e^{3x}\log\left(x+1\right)$.

8. $y''-2y'+2y=e^x\tan x$. **Ans:** $y=\left(c_1\cos x+c_2\sin x\right)e^x-e^x\cos x\log\left(\sec x+\tan x\right)$.

9. $\frac{d^2y}{dx^2}+a^2y=\text{cosec}\,ax$. **Ans:** $y=\left(c_1-\frac{x}{a}\right)\cos ax+\left[c_2+\left(1/a^2\right)\log\sin ax\right]\sin ax$.

10. $\left(D^2-2D+1\right)y=e^x\log x$ **Ans:** $y=\left(c_1+c_2x\right)e^x+\frac{x^2}{4}e^x\left(2\log x-3\right)$.

LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

The linear differential equations with variable coefficients can be reduced to linear differential equations with constant coefficients by suitable substitutions.

Cauchy's linear differential equation:

An equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + k_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + k_{n-1} x \frac{dy}{dx} + k_n y = X \quad \text{.............(1)}$$

Using the substitution $x = e^t$ in equation (1) we get $xDy = D_1 y, x^2 D^2 y = D_1(D_1 - 1)y$,.

Similarly $x^3 D^3 y = D_1(D_1 - 1)(D_1 - 2)y$ and so on, where $D = d/dx, D_1 = d/dt$

After making these substitutions in equation (1), there results a linear equation with constant coefficients which can be solved as before.

Problems

1. $x \frac{d^2 y}{dx^2} - \frac{2y}{x} = x + \frac{1}{x^2}$ **Ans:** $y = c_1 x^2 + \frac{c_2}{x} + \frac{1}{3}\left(x^2 - \frac{1}{x}\right)\log x$
2. $x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = 4x - 6$ **Ans:** $y = c_1 x^2 + c_2 x^3 + 2x - 1$
3. $x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + y = \frac{1}{(1-x)^2}$. **Ans:** $y = \left\{c_1 + c_2 \log x + \log \frac{x}{x-1}\right\} \frac{1}{x}$
4. $x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^{e^x}$ **Ans:** $y = \frac{c_1}{x} + \frac{(c_2 + e^x)}{x^2}$
5. $x^2 y'' + xy' + y = \log(x) \sin(\log x)$

   Ans. y=$c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{1}{4}(\log x)^2 \cos(\log x) + \frac{1}{4}\log(\log x)\sin(\log x)$
6. $x^2 y'' - 3xy' + 5y = x^2 \sin(\log x)$ **Ans.** $y = x^2(c_1 \cos\log x + c_2 \sin logx) - \log x.\frac{x^2}{2}.\cos\log x$

Legendre's linear differential equation:

An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + k_2 (ax+b)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + k_{n-1}(ax+b)\frac{dy}{dx} + k_n y = X \quad \text{.......(2)}$$

is called the Legendre's linear differential equation. Using the substitution $ax+b = e^t$ in equation (1) we get

$(ax+b)Dy = aD_1 y$, $(ax+b)^2 D^2 y = a^2 D_1(D_1 - 1)y$, $(ax+b)^3 D^3 y = a^3 D_1(D_1 - 1)(D_1 - 2)y$ and so on where $D = \frac{d}{dx}, D_1 = \frac{d}{dt}$. After making these substitutions in (1), there results a linear equation with constant coefficients, which can be solved as before.

Problems

7. $(x+2)^2 y'' + 3(x+2)y' - 3y = 0$ **Ans:** $y = c_1(x+2) + c_2(x+2)^{-3}$
8. $(2x+3)^2 y'' - (2x+3)y' - 12y = 6x$

   Ans: $y = c_1(2x+3)^a + c_2(2x+3)^b - \frac{3}{14}(2x+3) + \frac{3}{4}$, where $a, b = \frac{3 \pm \sqrt{57}}{4}$
9. $(1+x)^2 \frac{d^2 y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin[2\log(1+x)]$

   Ans: $y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - \frac{1}{3}\sin[2\log(1+x)]$

10. $(x-1)^3\frac{d^3y}{dx^3}+2(x-1)^2\frac{d^2y}{dx^2}-4(x-1)\frac{dy}{dx}+4y=4\log(x-1)$

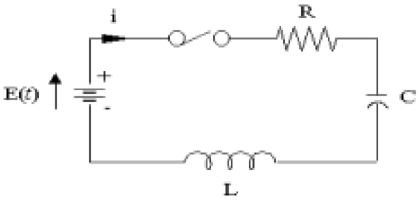
Ans: $y=c_1(x-1)+c_2(x-1)^2+c_3(x-1)^{-3}+\log(x+1)+1$.

11. $(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1$

Ans: $y=c_1(3x+2)^{1/3}+c_2(3x+2)^{-1}+\frac{1}{27}\left[\frac{1}{15}(3x+2)^2+\frac{1}{4}(3x+2)-7\right]$

APPLICATIONS

LRC- series circuit



Kirchhoff's voltage law (KVL): The algebraic sum of all the instantaneous voltage drops across the three elements inductor, resistor and capacitance is equal to the resultant electromotive force (external source) in the circuit. Thus the LRC-circuit is modeled as:

$$L\frac{dI}{dt}+RI+\frac{1}{c}Q=E(t).$$

Since $I=\frac{dQ}{dt}$ the above equation can be written as

$$L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{c}Q=E(t) \quad \text{.........(1)}$$

Differentiating (10) w.r.t. 't', we obtain

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{c}I=\frac{dE}{dt} \quad \text{................(2)}$$

Thus the charge Q and current I at any time in the LRC-circuit are obtained by solving the equations (1) and (2) respectively.

Alternately, If equation (1) is solved, then current can be obtained by $I=\frac{dQ}{dt}$ and if equation (2) is solved then Charge can be obtained by integrating the solution of (2) with respect to t.

Problems

1. A condenser of capacity C discharged through an inductance L and resistance R in series and the charge q at time t satisfies the equation $L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=0$. Given that L=0.25henries, R=250 ohms, $C=2\times10^{-6}$ farads and that when $t=0$, charge q is 0.002 coulombs and the current $\frac{dq}{dt}=0$, obtain the value of q in terms of t.
2. Determine Q and i in the LRC-circuit with $L=0.5H, R=6\Omega, C=0.02F$, $E(t)=24\sin(10t)$ and initial conditions $Q=i=0$ at t=0.
3. Find the charge on the capacitor in an LRC-series circuit at $t=0.01s$ when $L=0.05H$, $R=2\Omega$, $C=0.01F$, $E(t)=0V$, $q(0)=5C$ and $i(0)=0A$. Determine the first time when the charge on the capacitor is zero.
4. Find the charge on the capacitor in an LRC-series circuit when $L=\frac{1}{4}H$, $R=20\Omega$, $C=\frac{1}{300}F$, $E(t)=0V$, $q(0)=4C$ and $i(0)=0A$. Is the charge on the capacitor ever equal to zero?
5. Find the charge on the capacitor in an LRC-series circuit when $L=1H$, $R=100\Omega$, $C=0.0004F$, $E(t)=30V$, $q(0)=0C$ and $i(0)=2A$. Determine the first time when the charge on the capacitor is zero.

6. Find the steady-state charge and the steady-state current in an LRC-series circuit when $L = 1H$, $R = 2\Omega$, $C = 0.25F$, $E(t) = 50\cos(t)V$.

7. Find the steady-state charge and the steady-state current in an LRC-series circuit when $L = \frac{1}{2}H$, $R = 20\Omega$, $C = 0.001F$, $E(t) = 100\sin(60t) + 200\cos(40t)V$.

8. Find the steady-state charge and the steady-state current in an LRC-series circuit when $L = \frac{1}{2}H$, $R = 20\Omega$, $C = 0.001F$, $E(t) = 100\sin(60t)$.

9. Find the current in the RLC circuit connected in series for the given data. Assume zero initial current and charge
   a. $R = 200\Omega$, $L = 0.1H$, $C = 0.006F$ and $E = te^{-t}$ *volts*.
   b. $R = 400\Omega$, $L = 0.12H$, $C = 0.04F$ and $E = 120\sin(20t)$ *volts*.

<u>Mass-Spring Mechanical System</u>

Observations:

i) Newton's second law: Net force = mass × acceleration

ii) Hooke's Law: The tension of an elastic string (or a spring) is proportional to extension of the string (or a spring) beyond its natural length. i.e., $T = \frac{\lambda e}{l}$, where *e* is the extension beyond the natural length *l* and λ is the modulus of elasticity.

The differential equation describing the motion of the mass is obtained by Newton's second law

$$\text{as } m\frac{d^2x}{dt^2} = mg - k(e+x) - c\frac{dx}{dt} + F(t)$$

where

$mg =$ A gravitational force.

$k(e+x) =$ Spring restoring force due to displacement of the spring from equilibrium (rest) position.

$c\frac{dx}{dt} =$ A frictional (damping or resistance) force of the medium which opposes the motion.

$F(t) =$ External force.

When the mass is at rest, we get $mg = ke$ as all other forces are non-existant.

Problems

1. A mass weighing 2 pounds stretches a spring 6 inches. At $t = 0$ the mass is released from a point 8 inches below the equilibrium position with an upward velocity of $\frac{4}{3}\, ft\, s^{-1}$. Find the equation of motion.

2. A mass weighing 24 pounds, attached to a spring, stretches it 4 inches. Initially, the mass is released from rest from a point 3 inches above the equilibrium position. Find the equation of motion.

3. A mass weighing 24 pounds, attached to a spring, stretches it 4 inches. Initially, the mass is released from the equilibrium position with an initial downward velocity of $2\, ft\, s^{-1}$. Find the equation of motion.

4. A mass weighing 20 pounds stretches the spring 6 inches. The mass is initially released from rest from a point 6 inches below the equilibrium position. Find the equation of motion.
5. A force of 400 N stretches the spring 2 m. A mass of 50 kg is attached to the end of the spring and is initially released from the equilibrium position with an upward velocity of $10ms^{-1}$. Find the equation of motion.
6. An 8 pound weight stretches a spring 2 feet. Assuming that a damping force is equal to two time the instantaneous velocity acts on the system, determine the equation of motion if the weight is released from the equilibrium position with an upward velocity of 3 fs^{-1}.
7. A 4 foot spring measures 8 feet long after a mass weighing 8 pounds is attached to it. The medium through which the mass is offers damping force is numerically equal to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released with a downward velocity of 5 fts^{-1}.
8. A 16 pound weight is attached to a 5 foot spring. At equilibrium position the spring measures 8.2 feet. If the weight is pushed up and released from rest at a point 2 feet above the equilibrium position, find the displacement if it is further known that the surrounding medium offers a resistance equal to the instantaneous velocity.
9. A mass weighing 16 pounds stretches the spring $\frac{8}{3}$ ft. The mass is initially released from rest from a point 2ft below the equilibrium position, and the subsequent motion takes place in a medium that offers a damping force numerically equal to one-half the instantaneous velocity. Find the equation of motion if the mass driven by an external force equal to $f(t)=10\cos(3t)$.
10. A mass of 1 slug is attached to a spring whose spring constant is $5\ lb/ft$. Initially the mass is released one foot below the equilibrium position with a downward velocity of $5\ ft\,s^{-1}$, and the subsequent motion takes place in a medium that offers a damping force numerically equal to two times the instantaneous velocity. Find the equation of motion if the mass driven by an external force equal to $f(t)=12\cos(2t)+3\sin(2t)$.
11. A 32 lb weight is suspended from a coil spring stretches the spring to $2\,ft$. The weight is then pulled down 6 inches from the equilibrium position and released at $t=0$. Find the motion of the weight and determine the nature of motion if the resistance of the medium is $4\dot{x}$.
12. If a weight 6 lbs hangs from a spring with constant k=12 and no damping force exists, find the motion of weight when an external force 3cos8t acts. Initially $x=0$ and $\dot{x}=0$.
13. If weight $\omega=16lb$, spring constant $k=10\ lb/ft$, damping force $2\dot{x}$, external force $F(t)$ is $5\cos 2t$, find the motion of the weight given $x(0)=\dot{x}(0)=0$.
14. Determine the displacement of a body of weight 10kg attached to spring given that 20kg weight will stretch the spring 10 cm.
15. A 32 lb weight is suspended from a coil spring stretches the spring to 2 ft. The weight is then pulled down 6 inches from the equilibrium position and released at t=0. Find the motion of the weight and determine the nature of motion if the resistance of the medium is $8\dot{x}$.
16. A 16-pound weight is suspended from a spring, stretching it $8\,ft$. Then the weight is submerged in a fluid that imposes a drag of $6v$ pounds. The system is subjected to an external force $4\cos(3t)$. Find the solution satisfying. a) $y(0)=6$ and $v(0)=0$. b) $y(0)=0$ and $v(0)=6$.