MATHEMATICAL FOUNDATION FOR CIVIL ENGINEERING -1
And
MATHEMATICAL FOUNDATION FOR MECHANICAL ENGINEERING STREAM-1
And
MATHEMATICAL FOUNDATION FOR COMPUTER SCIENCE STREAM -1
And
MATHEMATICAL FOUNDATION FOR ELECTRICAL STREAM -1

UNIT 3: ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

Differential equations:
Any equation which involves derivatives, dependent variables and independent variables is called a differential equation.

Classification	
Ordinary differential equations	Linear differential equations Nonlinear differential equations
Partial differential equations	Linear, semi linear, quasi linear …

Solution of a differential equation: A relation between x and y which satisfies the given differential equation.

General Solution: A solution which contains arbitrary constants or functions.

Solution of first order ODE: Geometrically, the solution of a first order ODE represents a family of curves.

Linear Ordinary differential equations of 1st order

An equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ is called a linear first order differential equation.

The solution is given by $y(IF) = \int (IF)Q + c$ where $IF = e^{\int Pdx}$ and c is the constant of integration.

Bernoulli's Equation

The equation of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ is called the Bernoulli's equation.

It can be reduced to a linear differential equation by dividing throughout by y^n, to get $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$ and substituting $y^{1-n} = z$.

I. Solve the following differential equations:

1. $x\frac{dy}{dx} + y = x^3y^6$.
2. $2xy' = 10x^3y^5 + y$.
3. $y' + \frac{y}{2x} = \frac{x}{y^3}, y(1) = 2$.
4. $r\sin\theta - \cos\theta\frac{dr}{d\theta} = r^2$.
5. $(y\log x - 2)ydx - xdy = 0$.
6. $y^2y' - y^3\tan x - \sin x\cos^2 x = 0$.
7. $(x^3y^2 + xy)dx = dy$

8. $\dfrac{dy}{dx}=\dfrac{x^2+y^2+1}{2xy}$.
9. $\cos x dy = y(\sin x - y)dx$.
10. $\dfrac{dy}{dx}=\dfrac{y}{x+\sqrt{xy}}$.
11. $xy\left(1+xy^2\right)\dfrac{dy}{dx}=1$.
12. $\dfrac{dz}{dx}+\dfrac{z}{x}\log z=\dfrac{z}{x}(\log z)^2$.
13. $3x\left(1-x^2\right)y^2y'+\left(2x^2-1\right)y^3=ax^3$.
14. Suppose a student carrying a flu virus returns to an isolated college of 1000 students. It is assumed that the rate at which the virus spreads is proportional not only to the number x of the infected students but also to the number of students not infected. The equation is given by $\dfrac{dx}{dt}=kx(1000-x)$, $x(0)=1$. Determine the number of infected students after 6 days if it is further observed that after 4 days, $x(4)=50$ by the treating the equation as Bernoulli's equation.

EXACT DIFFERENTIAL EQUATIONS

A differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is said to be exact if $M(x,y)dx+N(x,y)dy$ is the exact differential of some function $u(x,y)$ i.e. $du=Mdx+Ndy$. Its solution, therefore is $u(x,y)=c$.

The **necessary and sufficient condition** for the differential equation $Mdx+Ndy=0$ to be exact is $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$.

The solution is given by $\displaystyle\int_{(y\text{ const})} Mdx+\int(\text{terms of } N \text{ without } x)dy=C$.

II. Solve the following differential equations:

1. $\left(x^4-2xy^2+y^4\right)dx-\left(2x^2y-4xy^3+\sin y\right)dy=0$.
2. $\left(5x^4+3x^2y^2-2xy^3\right)dx+\left(2x^3y-3x^2y^2-5y^4\right)dy=0$.
3. $\dfrac{2x}{y^3}dx+\dfrac{y^2-3x^2}{y^4}dy=0$.
4. $y'=\dfrac{y-2x}{2y-x}$, y(1)=2.
5. $\left[y\left(1+\dfrac{1}{x}\right)+\cos y\right]dx+\left(x+\log x-x\sin y\right)dy=0$.
6. $\left(2xy+y-\tan y\right)dx+\left(x^2-x\tan^2 y+\sec^2 y\right)dy=0$.
7. $\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$.
8. $\left(\sec x\tan x\tan y-e^x\right)dx+\sec x\sec^2 y dy=0$.
9. $\left(y\sin 2x\right)dx-\left(1+y^2+\cos^2 x\right)dy=0$.
10. $\left(3x^2y+\dfrac{y}{x}\right)dx+\left(x^3+\log x\right)dy=0$.
11. $\left(\cos x-x\cos y\right)dy-\left(\sin y+y\sin x\right)dx=0$.

12. $\left(\sin x\sin y - xe^{y}\right)dy = \left(e^{y} + \cos x\cos y\right)dx$.
13. $\sin x\cosh y dx - \cos x\sinh y dy = 0$, $y(0) = 3$.
14. $ye^{xy}dx + \left(xe^{xy} + 2y\right)dy = 0$.
15. $\left(\dfrac{y}{(x+y)^2} - 1\right)dx + \left(1 - \dfrac{x}{(x+y)^2}\right)dy = 0$.

EQUATIONS REDUCIBLE TO EXACT DIFFERENTIAL EQUATIONS

Case 1: $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$ or k then the integrating factor is $e^{\int f(x)dx}$ or $e^{\int kdx}$.

Case 2: $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{-M} = f(y)$ or k then the integrating factor is $e^{\int f(y)dy}$ or $e^{\int kdy}$.

III. Solve the following differential equations:

1. $\left(xy^3 + y\right)dx + 2\left(x^2y^2 + x + y^4\right)dy = 0$.
2. $y\log y dx + \left(x - \log y\right)dy = 0$.
3. $\left(xy^2 - e^{1/x^3}\right)dx - x^2 y dy = 0$
4. $y\left(2x^2 - xy + 1\right)dx + \left(x - y\right)dy = 0$.
5. $\left(xy^3 + y\right)dx + 2\left(x^2y^2 + x + y^4\right)dy = 0$.
6. $\left(3x^2y^4 + 2xy\right)dx + \left(2x^3y^3 - x^2\right)dy = 0$.
7. $y\left(x + y\right)dx + \left(x + 2y - 1\right)dy = 0$.
8. $2xydy - \left(x^2 + y^2 + 1\right)dx = 0$.
9. $3\left(x^2 + y^2\right)dx + x\left(x^2 + 3y^2 + 6y\right)dy = 0$.
10. $2xydx + \left(y^2 - x^2\right)dy = 0$.

IV. Orthogonal trajectories

1. Find the orthogonal trajectories of the family of:
   a) $y^2 = 4ax$.
   b) $x^2 - y^2 = c$.
   c) $y = c\left(\sec x + \tan x\right)$.
   d) $y = ax^2$.
   e) $x^2 - y^2 = cx$.
2. Find constant 'e' such that $y^3 = c_1 x$ and $x^2 + ey^2 = c_2$ are orthogonal to each other.
3. Find the value of constant d such that the parabolas $y = c_1x^2 + d$ are the orthogonal trajectories of the family of ellipses $x^2 + 2y^2 - y = c_2$.
4. The electric lines of force of two opposite charges of the same strength at $(1,0)$ and $(-1,0)$ are circles (through these points) of the form $x^2 + y^2 - ay = 1$. Find their equipotential lines (orthogonal trajectories).

5. Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$, where λ is the parameter.
6. Given $y=ce^{-2x}+3x$, find that member of the orthogonal trajectory which passes through point (0, 3).
7. The electric field between two concentric cylinders, the equipotential lines (lines of constant potential) are circles given by $U(x,y)=x^2+y^2=\text{constant}(\text{volts})$, what are their orthogonal trajectories (the curves of electric force)?
8. If the streamlines of the flow are (paths of the particles of the fluid) in the channel are $\psi(x,y)=xy=\text{constant}$, what are their orthogonal trajectories (called equipotential lines)?
9. If the isotherms (curves of constant temperature) in a body are $T(x,y)=2x^2+y^2=C$, a constant, what are their orthogonal trajectories (curves along which heat will be flowing in regions free of heat sources or sinks and filled with homogeneous material)?
10. Isotherms of a lamina in the xy-plane are defined by $x^2+3y^2=c$. What are their orthogonal trajectories (flux lines). Then determine the isotherm and the flux line passing through

    a) $(0,2)$ b) $(-1,1)$ c) $(2,0)$.
11. Show that the family of curves are self-orthogonal:
    a) $x^2=4a(y+a)$.
    b) Confocal conics, $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,\lambda$ being the parameter.
    c) $y^2=4a(x+a)$.
12. Find the orthogonal trajectories (O.T.) of each of the following family of curves:
    a) $r(1+2\cos\theta)=k$.
    b) $r=a(1+\cos\theta)$.
    c) $r^2=a^2\cos 2\theta$.
    d) $r=\frac{2a}{(1+\cos\theta)}$.
    e) $r=2a(\sin\theta+\cos\theta)$.
    f) $\left(r+\frac{k^2}{r}\right)cos\,\theta=d$, d being a parameter
    g) $r=a(1+\sin^2\theta)$.
    h) $r=a(\sec\theta+\tan\theta)$.

V. Chemical reactions (Mixing problems) [FOR Civil and Mechanical Engineering]

1. A tank is initially filled with 100 gallons of salt solution containing 1 lb of salt per gallon. Fresh brine containing 2 lb per gallon of salt runs into the tank at the rate of 5 gallons per minute and the mixture, assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt at any time, and determine how long will it take for this amount to reach 150 lb.
2. A tank initially contains 50 gallons of fresh water. Brine, containing 2 pounds per gallon of salt, flows into the tank at the rate of 2 gallons per minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it take for the quantity of salt in the tank to increase from 40 to 80 pounds?
3. A tank contains 1000 gallons of brine in which 500 lb of salt are dissolved. Fresh water runs into the tank at the rate of 10 gallons per minute and the mixture is kept uniform by stirring, runs out at the same rate. How long will it be before only 50 lb of salt is left in the tank?
4. Suppose a large mixing tank initially holds 300 gallons of brine. Another brine solution is pumped into the tank at a rate of 3gallons per minute; the concentration of the salt in this inflow is 2 lb/gal of salt. When the solution is stirred well, it is pumped out at the same rate. If there were 50 lb of salt dissolved initially in the 300 gallons, how much salt is there in the long time?
5. A tank contains 200 liters of fluid in which 30 grams of salt is dissolved. Brine containing 1 gram of salt per liter is then pumped into the tank at a rate of 4 liters per minute; the well mixed solution is pumped out at the same rate. Find the number of grams of salt in the tank at time t.

VI. Growth and Decay [FOR COMPUTER SCIENCE STREAM]

1. A culture initially has P_0 number of bacteria. At $t=1$ hour, the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.
2. A breeder reactor converts relatively stable uranium-238 into isotope plutonium-239. After 15 years, it is determined that 0.043% of the initial amount A_0 of the plutonium has disintegrated. Find the half-life of this isotope if the disintegration is proportional to amount remaining.
3. The population of a community is known to increase at a rate proportional to the number of people present at time t. If an initial population P_0 has doubled in 5 years, how long will it take to triple? To quadruple?
4. The population of a community is known to increase at a rate proportional to the number of people present at time t. An initial population P_0 has doubled in 5 years. Suppose it is known that the population of the community is 10000 after 3 years. What will the population be in 10 years? How fast is the population growing at $t=10$?
5. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at a time t and has a half-life of 3.3 hours. If 1 g of this isotope is present initially, how long will it take for 90% of the lead to decay?

6. Initially 100 mg of a radioactive substance was present. After 6 hours the mass has decreased by 3%. If the rate of decay is proportional to amount of the substance present at time t, find the amount remaining after 24 hours.

VII. LR Circuits [FOR ELECTRICAL STREAM]

1. A generator having emf 100 $volts$ is connected in series with a 10 ohm resistor and an inductor of 2 $henries$. If the switch is closed at a time $t=0$, determine the current time $t>0$.

2. Find the $I(t)$ in the RL – circuit with $R=10\Omega$, $L=100H$ and $E=40V$ when $0\le t\le 100$ and $E=0$ when $t>100$ and $I(0)=4$.

3. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current build up a rate given by $L\frac{di}{dt}+Ri=E$. Find i as a function of t. How long will it be, before the current has reached one-half its initial value if $E=6V$, $R=100\Omega$ and $L=0.1H$?

4. Obtain the differential equation for the current i in an electrical circuit containing an inductance L and a resistor R in series and acted by an electromotive force $E\sin(\omega t)$. Find the value of the current at any time t, if initially there is no current in the circuit.

5. Find $I(t)$ in an RL – circuit with $E=10V$, $R=5\Omega$, $L=(10-t)H$ when $0\le t\le 10s$ and $L=0$ when $t>10s$ and $I(0)=0$.