Unit 3: Vector spaces and Linear transformations

Course Code: 23MA2BSMCS/23MA2BSMES

I. Vector spaces $(V, +, \cdot)$ over a field K.

A nonempty set V is said to be a vector space over field K if

- $(V, +)$ is an abelian group.
- $k \cdot u \in V$ for all $u \in V$ and $k \in F$
- $k \cdot (u + v) = k \cdot u + k \cdot v$
- $(k_1 + k_2) \cdot u = k_1 \cdot u + k_2 \cdot u$
- $(k_1 k_2) \cdot u = k_1 \cdot (k_2 \cdot u)$
- $1 \cdot u = u$.

II. Subspaces

Non-empty subsets of a vector space which are vector space over the same field under the same vector addition and scalar multiplication are called subspaces.

A nonempty set W is a subspace of a vector space V over a field K if

- Zero vector is in W.
- $u + v \in W$ and $ku \in W$ for all $u, v \in W$ and $k \in K$ or $au + bv \in W$ for $u, v \in W$ and $a, b \in K$.

Intersection of any two subspaces of V is also a subspace of V.

Problems

1. Which of the following subsets of $\mathbb{R}^2$ with the usual operations of vector addition and scalar multiplication are subspaces?
   a) W_1 is the set of all vectors of the form (x, y), where $x \geq 0$.
   b) W_2 is the set of all vectors of the form (x, y), where $x \geq 0$ and $y \geq 0$.
   c) W_3 is the set of all vectors of the form (x, y), where $x = 0$.
   d) W_4 is the set of all vectors of the form (x, y), where $xy = 0$.
   e) W_5 is the set of vectors of the form $(3s, 2 + 5s)$ for $s \in \mathbb{R}$
2. Determine whether or not W is a subspace of R^3 where W consists of all vectors of the form (a, b, c)in R^3such that:
   a. $a = 3b$.
   b. $a \leq b \leq c$.
   c. $ab = 0$.
   d. $a + b + c = 0$.
   e. $b = a^2$.
   f. $a = 2b = 3c$.
   g. $a + b + c = 3$
   h. $a \geq 0$.
   i. $a^2 + b^2 + c^2 \leq 1$.
3. Determine whether or not $W = \{(\alpha, 4\alpha, 5\alpha): \alpha \text{ is any scalar}\}$ is a subspace of $\mathbb{R}^3$.
4. Determine whether or not the subset of matrices of the form $\begin{bmatrix} a & a^2 \\ b & b^2 \end{bmatrix}$ is a subspace.

III. Linear combination

A vector v is said to be the linear transformation of elements of set S if $v = c_1 s_1 + c_2 s_2 + c_3 s_3 + \dots\dots + c_n s_n$ for all s_i in S and $c_i$in F.

IV. Spanning sets or Linear span

Consider a subset $S = \{u_1, u_2, \cdots u_m\}$ of a vector space V over a field F. The collection of all possible linear combinations of vectors in S is called the linear span of S denoted by $span(S) = span(u_i)$. $span(S)$ is a subspace of V.

PROBLEMS

1. Verify whether v in $\mathbb{R}^3$ is a linear combination of the vectors u_1, u_2 and u_3:
   Or show that the given v is in Span(S), where $S = \{u_1, u_2, \cdots u_m\}$
   a. $v = (1,-2,5)$, $u_1 = (1,1,1)$, $u_2 = (1,2,3)$ and $u_3 = (2,-1,1)$.
   b. $v = (2,-5,3)$, $u_1 = (1,-3,2)$, $u_2 = (2,-4,-1)$ and $u_3 = (1,-5,7)$.
   c. $v = (4,-9,2)$, $u_1 = (1,2,-1)$, $u_2(1,4,2)$, and $u_3 = (1,-3,2)$.
   d. $v = (1,3,2)$, $u_1 = (1,2,1)$, $u_2(2,6,5)$ and $u_3 = (1,7,8)$.
   e. $v = (1,4,6)$, $u_1 = (1,1,2)$, $u_2(2,3,5)$ and $u_3 = (3,5,8)$.

2. Find k such that $v = (1,-2,k)$ is a linear combination of $(3,0,-2)$ and $(2,-1,-5)$.

3. Express the polynomial $v = t^2 + 4t - 3$ in $P(t)$ as a linear combination of the polynomials $p_1 = t^2 - 2t + 5$, $p_2 = 2t^2 - 3t$ and $p_3 = t + 1$.

4. Express $M = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}$ as a linear combination of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ & $C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$.

5. Check whether the following polynomials are in the $span(p_1, p_2, p_3)$ where $p_1 = t^3 + t^2$, $p_2 = t^3 + t$ and $p_3 = t^2 + t$. i) $p(t) = t^3 + t^2 + t$ ii) $p(t) = 2t^3 - t$ iii) $p(t) = 2t^2 + 1$.

V. Linear dependence and independence:

Consider $S = \{u_1, u_2, \cdots u_m\}$ of a vector space V. The set S is said to be linearly dependent if there exists some non-zero scalars $c_1, c_2 \cdots c_m$ such that $c_1u_1 + c_2u_2 + \cdots c_mu_m = 0$.

If such non-zero scalars do not exist, then S is said to be linearly independent.

- Any set of vectors containing the zero vector is a linearly dependent set.
- Any set with more than 1 vector is linearly dependent if and only if, one of them is a linear combination of the other vectors.

PROBLEMS

1. Determine whether or not each of the following lists of vectors in $\mathbb{R}^3$are linearly dependent:
   a) $u_1 = (1,2,3)$, $u_2 = (0,0,0)$, $u_3 = (1,5,6)$.
   b) $u = (1,1,2)$, $v = (2,3,1)$ and $w = (4,5,5)$.
   c) $u_1 = (1,2,5)$, $u_2 = (1,3,1)$, $u_3 = (2,5,7)$ and $u_4 = (3,1,4)$.
   d) $u_1 = (1,2,5)$, $u_2 = (2,5,1)$and $u_3 = (1,5,2)$.

2. Determine whether the following vectors in $\mathbb{R}^4$ are linearly dependent or independent:
   a. $(1,2,-3,1), (3,7,1,-2), (1,3,7,-4)$.
   b. $(1,3,1,-2), (2,5,-1,3), (1,3,7,-2)$.

3. Determine whether the following polynomials u, v and w in $P(t)$ are linearly dependent or independent:
   a. $u = t^3 - 4t^2 + 3t + 3$, $v = t^3 + 2t^2 + 4t - 1$ and $w = 2t^3 - t^2 - 3t + 5$.
   b. $u = t^3 - 5t^2 - 2t + 3$, $v = t^3 - 4t^2 - 3t + 4$ and $w = 2t^3 - 17t^2 - 7t + 9$.

VI. Bases and dimension

A nonempty subset $B = \{u_1, u_2, \cdots u_n\}$of a vector space V over a field K is said to be a basis of V if

- B is a linearly independent set.
- $span(B) = V$.

The number of vectors in the basis is called the dimension of V denoted by $dim(V) = n$.

- Any basis of a vector space will have the same number of vectors.
- In an n -dimensional vector space, any set containing more than n vectors are linearly dependent.
- Any linearly independent set with less than n vectors cannot span the vector space V. But this set is a subset of some basis of V and hence can be extended to a basis of V.

PROBLEMS

1. Determine whether or not each of the following form a basis of $\mathbb{R}^3$:
   a. (1,1,1), (1,0,1).
   b. $(1,1,1)$, $(1,2,3)$ and $(2,-1,1)$.
2. Determine whether $(1,1,1,1)$, $(1,2,3,2)$, $(2,5,6,4)$ and $(2,6,8,5)$ form a basis of $\mathbb{R}^4$. If not find the dimension of the subspace they span.
3. Extend $\{u_1 = (1,1,1,1), u_2 = (2,2,3,4)\}$ to a basis of $\mathbb{R}^4$.
4. Find a subset of $S = \{u_1, u_2, u_3, u_4\}$ that gives a basis for $W = span(u_i)$ of $\mathbb{R}^5$ where:
   a. $u_1 = (1,1,1,2,3)$, $u_2 = (1,2,-1,-2,1)$, $u_3 = (3,5,-1,-2,5)$ and $u_4 = (1,2,1,-1,4)$.
   b. $u_1 = (1,-2,1,3,-1)$, $u_2 = (-2,4,-2,-6,2)$, $u_3 = (1,-3,1,2,1)$ and $u_4 = (3,-7,3,8,-1)$.
5. Find the basis and dimension of the subspace W spanned by:
   a. $u = t^3 + 2t^2 - 2t + 1$, $v = t^3 + 3t^2 - 3t + 4$ and $w = 2t^3 + t^2 - 7t - 7$ in $P_3(t)$.
   b. $u = t^3 + t^2 - 3t + 2$, $v = 2t^3 + t^2 + t - 4$ and $w = 4t^3 + 3t^2 - 5t + 2$ in $P_3(t)$.
   c. $A = \begin{bmatrix} 1 & -5 \\ -4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 2 & -4 \\ -5 & 7 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & -7 \\ -5 & 1 \end{bmatrix}$ in $V = M_{2,2}$.
6. Find the basis and dimension of the row space, column space and null space of the following matrices:
   Hint: rowsp(A) = span(R_i); colsp(A) = span(C_i); nullsp(A) =solution space of $AX = 0$.
   i. $\begin{bmatrix} 1 & 2 & -1 & -2 & 0 \\ 2 & 4 & -1 & 1 & 0 \\ 3 & 6 & -1 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \end{bmatrix}$
   ii. $\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & 1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$
   iii. $\begin{bmatrix} 1 & -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 5 & 1 \\ 0 & 1 & -1 & -1 & -3 \end{bmatrix}$
   iv. $\begin{bmatrix} 0 & 0 & 3 & 1 & 4 \\ 1 & 3 & 1 & 2 & 1 \\ 3 & 9 & 4 & 5 & 2 \\ 4 & 12 & 8 & 8 & 7 \end{bmatrix}$

VII. Linear transformation:

If U and V are vector spaces over the same field, then the mapping $L: U \to V$ is said to be a linear transformation if

a) $L(u + v) = L(u) + L(v) \forall u, v \in U$.
b) $L(k.u) = k.L(u) \forall k \in F, u \in U$.

Alternately, $L(k_1.u + k_2 v) = k_1.L(u) + k_2.L(v)$.

PROBLEMS

Check whether the following are linear transformations:

1. $T: R^2 \to R^2; T(a, b) = (2a, 3b)$
2. $T: M_{2\times2} \to M_{2\times2}; T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + c & 0 \\ 0 & c - d \end{bmatrix}$.
3. $T: M_{2\times2} \to M_{2\times2}; T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$

4. $T: P_2(t) \to P_3(t); T(f(t)) = t * f(t)$
5. $T: P_2(t) \to P_4(t); T(f(t)) = f(t^2)$
6. $T: V_1(R) \to V_3(R); T(x) = (x, x^2, x^3)$.
7. Let V be the vector space of differentiable functions $f: \mathbb{R} \to \mathbb{R}$ with basis S and let $D: V \to V$ be the differential operator defined by $D(f(t)) = \frac{d(f(t))}{dt}$. Is D a linear transformation.

Theorem: If $\boldsymbol{T}: \boldsymbol{U} \to \boldsymbol{V}$ is linear transformation and $B = \{u_1, u_2, u_3 \dots \dots u_n\}$ is a basis of U and $S = \{v_1, v_2, v_3 \dots \dots v_n\}$ any set of vectors in V, then there exists a unique linear transformation such that $T(u_1) = v_1, T(u_2) = v_2 \dots \dots T(u_n) = v_n$.
Meaning: Every element v in U can be written as $v = \sum_{i=1}^{n} c_i\, u_i$. So by definition, $T(v) = T[\sum_{i=1}^{n} c_i u_i] = \sum_{i=1}^{n} c_i T(u_i)$ i.e., image of every element is a linear combination of images of the basis vectors. Alternately, image of the basis spans $Range(T)$.

PROBLEMS
Find the linear transformation

8. $T: R^2 \to R^2$such that $T(1,2) = (3,0)$ and $T(2,1) = (1,2)$.
9. $T: R^2 \to R^2$such that $T(1,2) = (3,-1)$ and $T(0,1) = (2,1)$.
10. $T: R^2 \to R^2$that maps $(1,3)$ and $(1,4)$ into $(-2,5)$ and $(3,-1)$.
11. $T: R^2 \to R^3$such that $T(-1,0) = (-1,0,2)$ and $T(2,1) = (1,2,1)$.
12. $T: \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(1,1,1) = (1,1,1)$ and $T(1,2,3) = (-1,-2,-3)$. (Ans: Infinitely many possibilities. It depends on how we map the missing basis vector).
13. Find the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^4$ whose range space is spanned by $(1,2,0,-4)$ and $(2,0,-1,-3)$. (Hint: The 3rd basis vector should be mapped to zero vector as the given vectors are the basis of range space).

VIII. Matrix representation of linear transformation:

Let $T: U \to V$ be a linear transformation with $B = \{u_1, u_2, u_3 \dots u_n\}$ and $B' = \{v_1, v_2, v_3 \dots v_m\}$ as the basis of U and V respectively. The matrix of the linear transformation denoted by $[T]_{BB'}$ is given by coordinate vectors of $T(u_i)$ relative to B' written along the columns.
i.e., $[T]_{BB'} = [[T(u_1)]_{B'}[T(u_2)]_{B'} \cdots \cdots [T(u_n)]_{B'}]$ which is a $m \times n$ matrix.

PROBLEMS

1. Find the matrix of linear transformation $T: R^2 \to R^3$defined by $T\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a+b \\ a-b \\ 2b \end{bmatrix}$with respect to the basis $B = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$for $R^2$and $C = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$for R^3.
2. Find the matrix of linear transformation $T: R^3 \to R^2$defined by $T(x,y,z) = (2x, 4y + 5z)$ with respect to the basis$B = \{(1,0,0), (0,1,0), (0,0,1)\}$ for $R^3$and $C = \{(1,1), (0,1)\}$for R^2.
3. Find the matrix of linear transformation $T: V_2(R) \to V_2(R)$ defined by
4. $T(x,y) = (2x + 3y, 4x - 5y)$ with respect to basis $S = \{(1,2), (2,5)\}$.
5. Consider the matrix $A = \begin{bmatrix} 2 & 4 \\ 5 & 6 \end{bmatrix}$ which defines a linear operator on $\mathbb{R}^2$. Find the matrix of the linear transformation relative to the basis $S = \{u_1, u_2\} = \left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -7 \end{bmatrix}\right\}$.

6. Consider the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -1 & 0 \\ 1 & 4 & -2 \end{bmatrix}$ which defines a linear operator on $\mathbb{R}^2$. Find the matrix of the linear transformation relative to the basis $S = \{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.
7. Consider the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 7 & 4 \\ 1 & 4 & 3 \end{bmatrix}$ which defines a linear operator on $\mathbb{R}^2$. Find the matrix of the linear transformation relative to the basis $S = \{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.
8. Find the matrix of linear transformation $T: p_2(t) \to p_2(t)$ defined by $T(f(t)) = f(t^2)$ with respect to the basis $B = \{1, t, t^2\}$.
9. Let $T: R^3 \to R^2$ be the linear transformation such that $T(1,0,0) = (2,1)$,
10. $T(0,1,0) = (1,-1)$ and $T(1,1,0) = (0,3)$. Find the matrix representation of T with respect to the standard bases of R^3 and R^2.

IX. Rank-Nullity Theorem:

$\boldsymbol{Rank(T) = dim(Range(T))}$ and $\boldsymbol{Nullity(T) = dim(Ker(T))}$.

Theorem: If $T: U \to V$ is a linear transformation then $\text{Rank}(T) + \text{Nullity}(T) = \dim(U)$.

PROBLEMS

Find the basis for the range space$R(T)$, null space$N(T)$ for the following linear transformation and also verify rank-nullity theorem. Verify whether they one-one or onto.

1. $T: R^3 \to R^3$defined by $T(x, y, z) = (x + y, x - y, 2x + z)$.
2. $T: R^3 \to R^3$defined by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$.
3. $T: R^3 \to R^3$defined by $T(x, y, z) = (x + y, x - y, 2x + z)$.
4. $T: R^4 \to R^3$defined by $T(x, y, z, t) = (x - y + z + t, x + 2z + t, x + y + 3z - 3t)$.
5. $T: R^4 \to R^3$ where $T = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$.
6. $T: \mathbb{R}^5 \to \mathbb{R}^4$where $T = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}$.
7. $T: \mathbb{R}^5 \to \mathbb{R}^3$ given by $T(x, y, z, s, t) = (x + 2y + 2z + s + t, x + 2y + 3z + 2s - t, 3x + 6y + 8z + 5s - t)$