B.M.S. COLLEGE OF ENGINEERING, BENGALURU
DEPARTMENT OF MATHEMATICS

For the Course Code: 23MA1BSMCS
Course: Mathematical Foundation for Computer Science Stream – 1

Differential equations:
Any equation which involves derivatives, dependent variables and independent variables is called a differential equation.

Classification	
Ordinary differential equations	Linear differential equations Nonlinear differential equations
Partial differential equations	Linear, semi linear, quasi linear …

Solution of a differential equation: A relation between x and y which satisfies the given differential equation.

General Solution: A solution which contains arbitrary constants.

Solution of first order ODE: Geometrically, the solution of a first order ODE represents a family of curves.

Linear Ordinary differential equations of 1st order

An equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$ is called a linear first order differential equation.

The solution is given by $y(IF)=\int(IF)Q+c$ where $IF=e^{\int Pdx}$ and c is the constant of integration.

I. BERNOULLI'S EQUATION

The equation of the form $\frac{dy}{dx}+P(x)y=Q(x)y^n$ is called the Bernoulli's equation. It can be reduced to a linear differential equation by dividing throughout by y^n, to get $y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$ and substituting $y^{1-n}=z$.

<u>EXAMPLES:</u>

Solve the following differential equations:

1 $x\frac{dy}{dx}+y=x^3y^6$.

2 $y' + \frac{y}{2x} = \frac{x}{y^3}, y(1) = 2.$

3 $(y \log x - 2) y dx - x dy = 0.$

4 $(x^3 y^2 + xy) dx = dy.$

5 $\cos x dy = y(\sin x - y) dx.$

6 $r \sin\theta - \cos\theta \frac{dr}{d\theta} = r^2.$

7 $y^2 y' - y^3 \tan x - \sin x \cos^2 x = 0.$

8 $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}.$

9 $3y' + xy = xy^{-2}.$

10 $\frac{dy}{dx} = \frac{y}{x + \sqrt{xy}}.$

11 $\frac{dz}{dx} + \frac{z}{x} \log z = \frac{z}{x} (\log z)^2.$

12 $\frac{dy}{dx} = \frac{y}{x + \sqrt{(xy)}}$

13 $3x(1 - x^2) y^2 y' + (2x^2 - 1) y^3 = ax^3.$

14 Suppose a student carrying a flu virus returns to an isolated college of 1000 students. It is assumed that the rate at which the virus spreads is proportional not only to the number x of the infected students but also to the number of students not infected. The equation is given by $\frac{dx}{dt} = kx(1000 - x), x(0) = 1.$ Determine the number of infected students after 6 days if it is further observed that after 4 days, $x(4) = 50$ by the treating the equation as Bernoulli's equation.

II. EXACT DIFFERENTIAL EQUATIONS

A differential equation of the form $M(x, y) dx + N(x, y) dy = 0$ is said to be exact if $M(x, y) dx + N(x, y) dy$ is the exact differential of some function $u(x, y)$ i.e. $du = M dx + N dy.$ Its solution, therefore is $u(x, y) = c$.

The necessary and sufficient condition for the differential equation $M dx + N dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, and its solution is given by $\int_{(y \text{ constant})} M dx + \int (\text{terms of } N \text{ without } x) dy = c$.

EXAMPLES:

Solve the following differential equations:

1 $\left(x^4-2xy^2+y^4\right)dx-\left(2x^2y-4xy^3+\sin y\right)dy=0$.

2 $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\left(x+\log x-x\sin y\right)dy=0$.

3 $\left(2xy+y-\tan y\right)dx+\left(x^2-x\tan^2 y+\sec^2 y\right)dy=0$.

4 $\frac{2x}{y^3}dx+\frac{y^2-3x^2}{y^4}dy=0$.

5 $y'=\frac{y-2x}{2y-x}$, $y(1)=2$.

6 $\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$.

7 $\left(\sec x\tan x\tan y-e^x\right)dx+\sec x\sec^2 y\,dy=0$.

8 $\left(y\sin 2x\right)dx-\left(1+y^2+\cos^2 x\right)dy=0$.

9 $\left(\cos x-x\cos y\right)dy-\left(\sin y+y\sin x\right)dx=0$.

10 $\left(\frac{y}{(x+y)^2}-1\right)dx+\left(1-\frac{x}{(x+y)^2}\right)dy=0$.

11 $\sin x\cosh y\,dx-\cos x\sinh y\,dy=0$, $y(0)=3$.

III. DIFFERENTIAL EQUATIONS REDUCIBLE TO THE EXACT FORM

Type 1: In the differential equation $Mdx+Ndy=0$, if $\frac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)}{N}$ a function of x say $f(x)$, then $IF=e^{\int f(x)dx}$.

Type 2: In the differential equation $Mdx+Ndy=0$, if $\frac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)}{-M}$ a function of y say $F(y)$, then $IF=e^{\int F(y)dy}$.

EXAMPLES:

Solve the following differential equations by reducing it to exact differential equations

1. $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$
2. $y\,logy\,dx + (x - logy)dy = 0$
3. $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2ydy = 0$
4. $y(2x^2 - xy + 1)dx + (x - y)dy = 0$
5. $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0.$
6. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0.$
7. $y(x + y)dx + (x + 2y - 1)dy = 0.$
8. $2xydy - (x^2 + y^2 + 1)dx = 0.$
9. $3(x^2 + y^2)dx + x(x^2 + 3y^2 + 6y)dy = 0.$
10. $2xydx + (y^2 - x^2)dy = 0.$

IV. ORTHOGONAL TRAJECTORIES

In Cartesian Co-ordinates:

EXAMPLES:

1. Find the orthogonal trajectories of the family of:
   a) $y^2 = 4ax$. b) $y = c(secx + tanx)$. c) $y = ax^2$.

   d) $x^2 - y^2 = c$. e) $x^2 - y^2 = cx$.
2. Find constant 'e' such that $y^3 = c_1x \; and \; x^2 + ey^2 = c_2$ are orthogonal to each other.
3. Find the value of constant d such that the parabolas $y = c_1x^2 + d$ are the orthogonal trajectories of the family of ellipses $x^2 + 2y^2 - y = c_2$.
4. The electric lines of force of two opposite charges of the same strength at (1,0) and (−1,0) are circles (through these points) of the form $x^2 + y^2 - ay = 1$. Find their equipotential lines (orthogonal trajectories)
5. Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is the parameter.
6. Given $y = ce^{-2x} + 3x$, find that member of the orthogonal trajectory which passes through point (0, 3).
7. The electric field between two concentric cylinders, the equipotential lines (lines of constant potential) are circles given by $U(x, y) = x^2 + y^2 = constant(volts)$, what are their orthogonal trajectories (the curves of electric force)?
8. If the streamlines of the flow are (paths of the particles of the fluid) in the channel are $\psi(x, y) = xy = constant$, what their orthogonal trajectories (called equipotential lines) are?
9. If the isotherms (curves of constant temperature) in a body are $T(x, y) = 2x^2 + y^2 = C$, a constant, what are their orthogonal trajectories (curves along which heat will be flowing in regions free of heat sources or sinks and filled with homogeneous material)?
10. Isotherms of a lamina in the xy -plane are defined by $x^2 + 3y^2 = c$. What are their orthogonal trajectories (flux lines). Then determine the isotherm and the flux line passing through

a) (0,2) b) (-1,1) c) (2,0).

11. Show that the family of curves are self-orthogonal:
    a) $x^2 = 4a(y+a)$.
    b) Confocal conics, $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1, \lambda$ being the parameter.
    c) $y^2 = 4a(x+a)$.
12. Find the orthogonal trajectories (O.T.) of each of the following family of curves:

a) $r = \frac{2a}{(1+\cos\theta)}$

b) $r^2 = a^2 \cos 2\theta$

c) $r = 4a\sec\theta \cdot \tan\theta$

d) $r = a\left(1+\sin^2\theta\right)$

e) $r = a\sin\theta\tan\theta$

f) $r = a(\sec\theta + \tan\theta)$

g) $r^n \cos n\theta = a^n$

V. APPLICATIONS

Growth and Decay

EXAMPLES:

1. A culture initially has P_0 number of bacteria. At t=1 hour, the number of bacteria is measured to be $\frac{3}{2}p_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t , determine the time necessary for the number of bacteria to triple.
2. A breeder reactor converts relatively stable uranium-238 into isotope plutonium-239. After 15 years, it is determined that 0.043% of the initial amount A_0 of the plutonium has disintegrated. Find the half-life of this isotope if the disintegration is proportional to amount remaining.
3. The population of a community is known to increase at a rate proportional to the number of people present at time t . If an initial population P_0 has doubled in 5 years, how long will it take to triple? To quadruple?
4. The population of a community is known to increase at a rate proportional to the number of people present at time t . An initial population P_0 has doubled in 5 years. Suppose it is known that the population of the community is 10000 after 3 years. What will the population be in 10 years? How fast is the population growing at t =10 ?
5. The radioactive isotope of lead, Pb-209, decays at a rate proportional to the amount present at a time t and has a half-life of 3.3 hours. If 1 g of this isotope is present initially, how long will it take for 90% of the lead to decay?
6. Initially 100 mg of a radioactive substance was present. After 6 hours the mass has decreased by 3%. If the rate of decay is proportional to amount of the substance present at time t , find the amount remaining after 24 hours.