B.M.S. COLLEGE OF ENGINEERING, BENGALURU
DEPARTMENT OF MATHEMATICS

For the Course Code: 23MA1BSCEM
Course: Mathematical Foundation for Civil, Electrical and Mechanical Eng. Stream - 1

Differential equations:
Any equation which involves derivatives, dependent variables and independent variables is called a differential equation.

Classification	
Ordinary differential equations	Linear differential equations Nonlinear differential equations
Partial differential equations	Linear, semi linear, quasi linear …

Solution of a differential equation: A relation between x and y which satisfies the given differential equation.

General Solution: A solution which contains arbitrary constants.

Solution of first order ODE: Geometrically, the solution of a first order ODE represents a family of curves.

Linear Ordinary differential equations of 1st order

An equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$ is called a linear first order differential equation.

The solution is given by $y(IF)=\int(IF)Q+c$ where $IF=e^{\int Pdx}$ and c is the constant of integration.

I. BERNOULLI'S EQUATION

The equation of the form $\frac{dy}{dx}+P(x)y=Q(x)y^n$ is called the Bernoulli's equation. It can be reduced to a linear differential equation by dividing throughout by y^n, to get $y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$ and substituting $y^{1-n}=z$.

<u>**EXAMPLES:**</u>

Solve the following differential equations:

1 $x\frac{dy}{dx}+y=x^3y^6$.

2 $y'+\frac{y}{2x}=\frac{x}{y^3}, y(1)=2$.

3 $(y\log x-2)ydx-xdy=0.$

4 $(x^3y^2+xy)dx=dy.$

5 $\cos xdy=y(\sin x-y)dx.$

6 $r\sin\theta-\cos\theta\frac{dr}{d\theta}=r^2.$

7 $y^2y'-y^3\tan x-\sin x\cos^2 x=0.$

8 $\frac{dy}{dx}=\frac{x^2+y^2+1}{2xy}.$

9 $3y'+xy=xy^{-2}.$

10 $\frac{dy}{dx}=\frac{y}{x+\sqrt{xy}}.$

11 $\frac{dz}{dx}+\frac{z}{x}\log z=\frac{z}{x}(\log z)^2.$

12 $\frac{dy}{dx}=\frac{y}{x+\sqrt{(xy)}}$

13 $3x(1-x^2)y^2y'+(2x^2-1)y^3=ax^3.$

14 Suppose a student carrying a flu virus returns to an isolated college of 1000 students. It is assumed that the rate at which the virus spreads is proportional not only to the number x of the infected students but also to the number of students not infected. The equation is given by $\frac{dx}{dt}=kx(1000-x), x(0)=1.$ Determine the number of infected students after 6 days if it is further observed that after 4 days, $x(4)=50$ by the treating the equation as Bernoulli's equation.

II. EXACT DIFFERENTIAL EQUATIONS

A differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is said to be exact if $M(x,y)dx+N(x,y)dy$ is the exact differential of some function $u(x,y)$ i.e. $du=Mdx+Ndy.$ Its solution, therefore is $u(x,y)=c$.

The necessary and sufficient condition for the differential equation $Mdx+Ndy=0$ to be exact is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, and its solution is given by $\int_{(y\text{ constant})} Mdx+\int(\text{terms of } N \text{ without } x)dy=c$.

EXAMPLES:

Solve the following differential equations:

1 $(x^4-2xy^2+y^4)dx-(2x^2y-4xy^3+\sin y)dy=0.$

2 $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+\left(x+\log x-x\sin y\right)dy=0$.

3 $(2xy+y-\tan y)dx+\left(x^2-x\tan^2 y+\sec^2 y\right)dy=0$.

4 $\frac{2x}{y^3}dx+\frac{y^2-3x^2}{y^4}dy=0$.

5 $y'=\frac{y-2x}{2y-x}$, y(1)=2.

6 $\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$.

7 $\left(\sec x\tan x\tan y-e^x\right)dx+\sec x\sec^2 y dy=0$.

8 $(y\sin 2x)dx-\left(1+y^2+\cos^2 x\right)dy=0$.

9 $(\cos x-x\cos y)dy-(\sin y+y\sin x)dx=0$.

10 $\left(\frac{y}{(x+y)^2}-1\right)dx+\left(1-\frac{x}{(x+y)^2}\right)dy=0$.

11 $\sin x\cosh y dx-\cos x\sinh y dy=0$, $y(0)=3$.

III. DIFFERENTIAL EQUATIONS REDUCIBLE TO THE EXACT FORM

Type 1: In the differential equation $Mdx+Ndy=0$, if $\frac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)}{N}$ a function of x say $f(x)$, then $IF=e^{\int f(x)dx}$.

Type 2: In the differential equation $Mdx+Ndy=0$, if $\frac{\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)}{-M}$ a function of y say $F(y)$, then $IF=e^{\int F(y)dy}$.

<u>EXAMPLES:</u>

Solve the following differential equations by reducing it to exact differential equations

1. $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$
2. $y\,logy\,dx+(x-logy)dy=0$

3. $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2ydy = 0$
4. $y(2x^2 - xy + 1)dx + (x - y)dy = 0$
5. $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0.$
6. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0.$
7. $y(x + y)dx + (x + 2y - 1)dy = 0.$
8. $2xydy - (x^2 + y^2 + 1)dx = 0.$
9. $3(x^2 + y^2)dx + x(x^2 + 3y^2 + 6y)dy = 0.$
10. $2xydx + (y^2 - x^2)dy = 0.$

IV. ORTHOGONAL TRAJECTORIES

In Cartesian Co-ordinates:

EXAMPLES:

1. Find the orthogonal trajectories of the family of:
   a) $y^2 = 4ax.$ b) $y = c(secx + tanx).$ c) $y = ax^2.$
   d) $x^2 - y^2 = c.$ e) $x^2 - y^2 = cx.$
2. Find constant ‘e’ such that $y^3 = c_1x \; and \; x^2 + ey^2 = c_2$ are orthogonal to each other.
3. Find the value of constant d such that the parabolas $y = c_1x^2 + d$ are the orthogonal trajectories of the family of ellipses $x^2 + 2y^2 - y = c_2$.
4. The electric lines of force of two opposite charges of the same strength at (1,0) and (−1,0) are circles (through these points) of the form $x^2 + y^2 - ay = 1$. Find their equipotential lines (orthogonal trajectories)
5. Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$, where λ is the parameter.
6. Given $y = ce^{-2x} + 3x$, find that member of the orthogonal trajectory which passes through point (0, 3).
7. The electric field between two concentric cylinders, the equipotential lines (lines of constant potential) are circles given by $U(x,y) = x^2 + y^2 = constant(volts)$, what are their orthogonal trajectories (the curves of electric force)?
8. If the streamlines of the flow are (paths of the particles of the fluid) in the channel are $\psi(x,y) = xy = constant$, what their orthogonal trajectories (called equipotential lines) are?
9. If the isotherms (curves of constant temperature) in a body are $T(x,y) = 2x^2 + y^2 = C$, a constant, what are their orthogonal trajectories (curves along which heat will be flowing in regions free of heat sources or sinks and filled with homogeneous material)?
10. Isotherms of a lamina in the xy -plane are defined by $x^2 + 3y^2 = c$. What are their orthogonal trajectories (flux lines). Then determine the isotherm and the flux line passing through
    a) (0,2) b) (-1,1) c) (2,0).
11. Show that the family of curves are self-orthogonal:
    a) $x^2 = 4a(y + a).$
    b) Confocal conics, $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1, \lambda$ being the parameter.

c) $y^2 = 4a(x + a)$.

12. Find the orthogonal trajectories (O.T.) of each of the following family of curves:

a) $r = \frac{2a}{(1+\cos\theta)}$

b) $r^2 = a^2 \cos 2\theta$

c) $r = 4a \sec\theta \cdot \tan\theta$

d) $r = a\left(1+\sin^2\theta\right)$

e) $r = a\sin\theta\tan\theta$

f) $r = a(\sec\theta + \tan\theta)$

g) $r^n \cos n\theta = a^n$

V. APPLICATIONS

MIXING PROBLEM (For Electrical, Civil and Mechanical stream)

EXAMPLES:

1 A tank is initially filled with 100 gallons of salt solution containing 1 lb of salt per gallon. Fresh brine containing 2 lb per gallon of salt runs into the tank at the rate of 5 gallons per minute and the mixture, assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt at any time, and determine how long will it take for this amount to reach 150 lb.

2 Suppose a large mixing tank initially holds 300 gallons of brine. Another brine solution is pumped into the tank at a rate of 3gallons per minute; the concentration of the salt in this inflow is 2 lb/gal of salt. When the solution is stirred well, it is pumped out at the same rate. If there were 50 lb of salt dissolved initially in the 300 gallons, how much salt is there in the long time?

3 A tank contains 200 litres of fluid in which 30 grams of salt is dissolved. Brine containing 1 gram of salt per litre is then pumped into the tank at a rate of 4 L/min.; the well mixed solution is pumped out at the same rate. Find the amount of salt in the tank at time t.

4 A tank contains 1000 gallons of brine in which 500 lb of salt are dissolved. Fresh water runs into the tank at the rate of 10 gallons per minute and the mixture is kept uniform by stirring, runs out at the same rate. How long will it be before only 50 lb of salt is left in the tank?