Course: *Mathematical Foundation for Civil and Mechanical Engineering Stream- 2* ***(23MA2BSMCM)***

Unit 3: Partial Differential Equations

Formation of P.D.E

Type1: Elimination of arbitrary constants

Form the partial differential equation by eliminating arbitrary constants from following

1. $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ $\qquad$ $Ans: q = px + p^2$
2. $(x-a)^2 + (y-b)^2 + z^2 = 4$ $\qquad$ $Ans: z(p^2 + q^2 + 1) = 4$
3. $(x-a)^2 + (y-b)^2 = z$ $\qquad$ $Ans: p^2 + q^2 = 4z$
4. $z = ax + by + a^2 + b^2$ $\qquad$ $Ans: z = px + qy + p^2 + q^2$
5. $z = (x^2 + a^2)(y^2 + b^2)$ $\qquad$ $Ans: z = \frac{pq}{4xy}$
6. $z = axy + b$ $\qquad$ $Ans: px = qy$
7. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $\qquad$ $Ans: z^2 - z(px + qy) = c^2$
8. $z = a\log\left\{\frac{b(y-1)}{1-x}\right\}$ $\qquad$ $Ans: p(1-x) = q(y-1)$
9. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ $\qquad$ $Ans: 2z = (px + qy)$
10. $\log(az - 1) = x + ay + b$ $\qquad$ $Ans: qz - pq = 1$
11. $z = xy + y\sqrt{x^2 - a^2} + b$ $\qquad$ $Ans: (q - x)p = yq$
12. $z = Ae^{pt}sinpx$ $\qquad$ $Ans: (\partial^2 z/ \partial t^2) + (\partial^2 z/ \partial x^2) = 0$
13. Find the differential equation of all spheres of fixed radius having their centres in the xy-plane. $\qquad$ $Ans: z^2(p^2 + q^2 + 1) = R^2$
14. Find the differential equation of all spheres whose centres lie on the z-axis.
    $Ans: q(x + zp) = p(y + zq)$
15. Find the differential equation of all spheres of radius 3 units having their centres in the yz-plane. $\qquad$ $Ans: x^2(q^2 + 1) = p^2(9 - x^2)$

Type II: Elimination of arbitrary functions

If $u = F(v)$ OR $v = F(u)$ OR $F(u, v) = 0$, $u = u(x, y, z)$ and $v = v(x, y, z)$ then,

- Differentiate the equation partially w.r.t x and partially w.r.t y by chain rule.
- Eliminate Fx and Fy by expanding the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y} \end{vmatrix} = 0$
- The resultant is a first order linear P.D.E. of the form $Pp + Qq = R$

Form the partial differential equation by eliminating arbitrary functions from following

1. $z = (x + y)\phi(x^2 - y^2)$ $\quad Ans: z = y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}$
2. $F(x^2 + y^2, z - xy) = 0$ $\quad Ans: x(q - x) = y(p - y)$
3. $F(xy + z^2, x + y + z) = 0$ $\quad Ans: (1 + q)(y + 2zp) = (1 + p)(x + 2zp)$
4. $F(x^2 + y^2 + z^2, z^2 - 2xy) = 0$ $\quad Ans: x^2 - y^2 + z(x + y)(p - q) = 0$
5. $F(x + y + z, x^2 + y^2 - z^2) = 0$ $\quad Ans: (y - x) + yp - xq + z(p - q) = 0$
6. $F(x^2 + y^2, x^2 - z^2) = 0$ $\quad Ans: yzp - xzq = xy$
7. $F(x^2 + 2yz, y^2 + 2zx) = 0$ $\quad Ans: xy + x^2q + y^2p - z^2 - xzp - yzq = 0$
8. $x + y + z = f(x^2 + y^2 + z^2)$ $\quad Ans: (1 + p)(2y + 2zq) = (1 + q)(2x + 2zp)$
9. $xyz = f(x + y + z)$ $\quad Ans: (1 + q)(yz + xyp) = (1 + p)(xz + xyq)$
10. $F(ax + by + cz, x^2 + y^2 + z^2) = 0$ $\quad Ans: (a + cp)(y + zq) = (b + cq)(x + zp)$
11. $z = x^n f(y/x)$ $\quad Ans: xp + yq = nz$
12. $z = f(x^2 - y^2)$ $\quad Ans: py + qx = 0$
13. $z = f(\sin x + \cos y)$ $\quad Ans: psin(y) + qcos(x) = 0$
14. $z = e^{ax+by} f(ax - by)$ $\quad Ans: bp + aq = 2abz$
15. $z = y^2 + 2f(1/x + \ln y)$ $\quad Ans: qy + px^2 = 2y^2$
16. $z = x + y + f(xy)$ $\quad Ans: x - y = 0$
17. $z = f(xy/z)$ $\quad Ans: xp - yq = 0$
18. $lx + my + nz = f(x^2 + y^2 + z^2)$ $\quad Ans: (ny - mz)p + (lz - nx)q = mx - ly$
19. $z = f(x + at) + g(x - at)$ $\quad Ans: \frac{\partial^2 z}{\partial t^2} - a^2\frac{\partial^2 z}{\partial x^2} = 0$
20. $z = f(x + iy) + F(x - iy)$ $\quad Ans: \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

21. $z = f(x^2 - y) + g(x^2 + y)$ $Ans: z_{xx} - 4x^2 z_{yy} = \frac{z_x}{x}$

22. $z = \frac{1}{r}[f(r - at) + g(r + at)]$ $Ans: z_{tt} = a^2(z_{rr} + \frac{2}{r} z_r)$

Direct Integration Method for Non-homogeneous PDEs:

Solve by direct integration method:

1. $\frac{\partial^2 z}{\partial x \, \partial y} = 0$ $Ans: z = F(x) + G(y)$
2. $\frac{\partial^2 z}{\partial x^2} = \cos x$ $Ans: z = -cos(x) + c_1(y)x + c_2(y)$
3. $\frac{\partial^2 z}{\partial y^2} = \sin y$ $Ans: z = sin(y) + c_1(x)y + c_2(x)$
4. $\frac{\partial^2 z}{\partial x \, \partial y} = \frac{x}{y} + a$ $Ans: z = \frac{x^2}{2} \ln(y) + ayx + c_1(x) + c_2(y)$
5. $\frac{\partial^2 z}{\partial x \, \partial t} = e^{-t} \cos x$ $Ans: z = -\sin(x)\, e^{-t} + D(x) + C_1(t)$
6. $xy \frac{\partial^2 z}{\partial x \, \partial y} = 1$ $Ans: z = \ln|y| \ln|x| + c_2(x) + c_3(y)$
7. $\frac{\partial^2 z}{\partial y^2} = x^2 y^3$ $Ans: z = \frac{x^2 y^5}{20} + c_1(x)y + c_2(x)$
8. $\frac{\partial^2 z}{\partial x \, \partial y} = e^{x+y}$ $Ans: z = e^{x+y} + f(x) + g(y)$

Lagrange's Linear Partial Differential Equations

Working Rule to obtain the general solution

- Rewrite the equation in the standard form $Pp + Qq = R$.
- Form the Lagrange's auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
- Nature of solution to the simultaneous equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$:

$u(x, y, z) = c_1$ and $v(x, y, z) = c_2$

are said to be the complete solution of the system of simultaneous equations (provided u and v are linearly independent i.e., $u/v \neq$ constant.

Case 1: One of the variables is either absent or cancels out from the set of auxiliary equations.

Case 2: If $u(x, y, z) = c_1$ is known but $v(x, y, z) = c_2$ is not possible by case 1, then use $u(x, y, z) = c_1$ to get $v(x, y, z) = c_2$.

Case 3: Introducing Lagrange's multipliers P_1, Q1, R1, which are functions of (x, y, z) or constants, each fraction is equal to

$$\frac{P_1dx + Q_1dy + R_1dz}{P_1P + Q_1Q + R_1R}$$

If P_1, Q1, R1 are so chosen that $P_1P + Q_1Q + R_1R = 0$ then $P_1dx + Q_1dy + R_1dz = 0$ which can be integrated.

Problems: Solve the following PDE

1. $xp + yq = 3z$ $Ans: \phi\left(\frac{x}{y}, \frac{z}{x^3}\right) = 0$
2. $yzp - xzq = xy$ $Ans: \phi(x^2 + y^2, x^2 - z^2) = 0$
3. $p - q = \ln(x + y)$ $Ans: \phi(x + y, z - xln(x + y)) = 0$
4. $z(z^2 + xy)(px - qy) = x^4$ $Ans: \phi(xy, x^4 - z^4 - 2xyz^2) = 0$
5. $xzp + yzq = xy$ $Ans: \phi\left(\frac{x}{y},\ y - z\right) = 0$
6. $(z - y)p + (x - z)q = y - x$ $Ans: \phi(x + y + z, x^2 + y^2 + z^2) = 0$
7. $(y + zx)p - (x + yz)q = x^2 - y^2$ $Ans: \phi(x^2 + y^2 + z^2,\ x^2 + y^2 - z^2) = 0$
8. $y^2zp + x^2zq = xy^2$ $Ans: \phi(x^3 - y^3, x^2 - z^2) = 0$
9. $p\tan x + q\tan y = \tan z$ $Ans: \phi\left(\frac{\sin(x)}{\sin(y)}, \frac{\sin(x)}{\sin(z)}\right) = 0$
10. $2p + 3q = 1$ $Ans: \phi\left(x - \frac{2y}{3}, y - 3z\right) = 0$
11. $pyz + qzx = xy$ $Ans: \phi(x^2 - y^2,\ y^2 - z^2) = 0$
12. $x^2p + y^2q = (x + y)z$ $Ans: \phi\left(\frac{y-x}{xy}, \frac{z}{xy}\right) = 0$
13. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ $Ans: \phi(\sqrt{x} - \sqrt{y}, \sqrt{x} - \sqrt{z}) = 0$
14. $z(xp - yq) = y^2 - x^2$ $Ans: \phi(xy, z(x^2 - y^2) = 0$
15. $(mz - ny)p + (nx - lz)q = ly - mx$ $Ans: \phi\left(x + \frac{my}{l} + \frac{nz}{l}, lx + my + nz\right) = 0$
16. $x(y - z)p + y(z - x)q = z(x - y)$ $Ans: \phi(x + y + z, xyz) = 0$
17. $(y - z)p + (x - y)q = (z - x)$ $Ans: \phi(x + y + z, x^2 + y^2 + z^2) = 0$
18. $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$ $Ans: \phi\left(xyz, \frac{x-y}{y-z}\right) = 0$
19. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$ $Ans: \phi\left(xyz, \frac{x^2-y^2}{y^2-z^2}\right) = 0$
20. $(y^2 + z^2)p - xyp + zx = 0$ $Ans: \phi\left(\frac{y}{z}, x^2 + y^2 + z^2\right) = 0$
21. $(y + zx)p - (x + yz)q = x^2 - y^2$ $Ans: \phi(x^2 + y^2 + z^2, x - y + z) = 0$
22. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ $Ans: \phi(x^2 + y^2 + z^2, x^3 + y^3 + z^3 - 3xyz) = 0$

Method of Separation of Variables

Working Rule

- Consider the p.d.e $F\left(x, y, u(x,y), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \ldots\ldots\right) = 0$

- Assume the solution $u(x,y) = X(x)Y(y)$
- The pde reduces to the form $f(X, X', X'', \dots.) = g(Y, Y', Y'', \dots.)$ which is separable in X and Y.
- Equate the functions to a common constant k i.e., $f(X, X', X'', \dots.) = g(Y, Y', Y'', \dots.) = k$
- Solving the above O.D.E, gives the solution of considered P.D.E.

Solve the following equations by the method of Separation of variables

1. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ $\quad Ans: z(x,y) = \left[C_1 e^{(1+\sqrt{(1+\lambda)})x} + C_2 e^{(1-\sqrt{(1+\lambda)})x}\right] C_3 e^{-\lambda y}$
2. $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$ $\quad Ans: u(x,t) = 6e^{-3x-2t}$
3. $py^3 + qx^2 = 0$ $\quad Ans: z = F(4x^3 - 3y^4)$
4. $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = 0$ $\quad Ans: u = \phi\left(\frac{y-x}{xy}\right)$
5. $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$ given $u(0,y) = 8e^{-3y}$ $\quad Ans: u(x,y) = 8e^{-3(y+4x)}$
6. $u_{xx} - u_y = 0$ $\quad Ans: u(x,y) = e^x A(y) + B(x)$
7. $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$,, given $u(x,0) = 4e^{-x}$ $\quad Ans: u(x,y) = 4e^{-x+\frac{3y}{2}}$
8. $\mathrm{u_{xt}} = e^{-t}\cos x$ given $\mathrm{u(x,0)} = 0$ and $\frac{\partial u(0,t)}{\partial t} = 0$ $\quad Ans: u(x,t) = \sin(x)(1 - e^{-t})$

Heat and Wave equation

1. Derive the one-dimensional heat equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$, where $u(x,t)$ denotes the temperature at a point x at time t.

2. Derive the one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2}, = c^2\frac{\partial^2 u}{\partial x^2}$, where $u(x,t)$ denotes the displacement at a point x and at time $t > 0$.