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DEPARTMENT OF MATHEMATICS

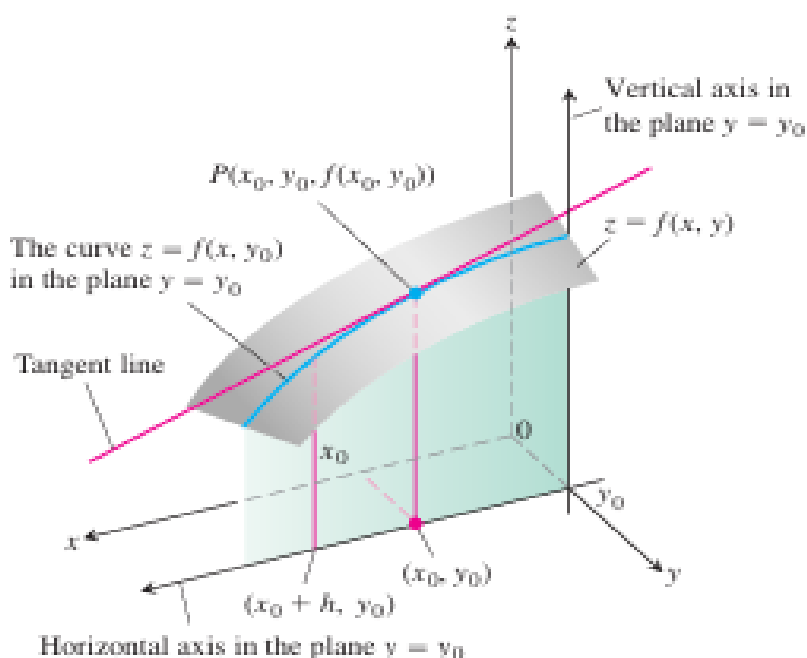
Course: *Mathematical Foundation for Computer Science Stream-1 (25MA1BSMCS)*

UNIT-2 MULTIVARIABLE CALCULUS

PARTIAL DERIVATIVES

Let $u = f(x, y)$ be a function of two variables x and y . If we keep y as constant and vary x alone, then u is a function of x only. The derivative of u with respect to x , treating y as constant is called the partial derivative of u w. r. t x and is denoted by one of the symbols $\frac{\partial u}{\partial x}$, u_x . Thus $\frac{\partial u}{\partial x} = u_x = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$. Similarly, $\frac{\partial u}{\partial y} = u_y = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$. Further u_x and u_y are also functions of x and y , so these can further be differentiated partially w. r. t. x and y . Thus $\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial x^2}$ or u_{xx} , $\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial x \partial y}$ or u_{xy} , $\frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial y \partial x}$ or u_{yx} and $\frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial y^2}$ or u_{yy} .

Geometrically, the slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at (x_0, y_0) .





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In general, if f is function of n independent variables say $x_1, x_2, \dots, x_n$ then the partial derivative of f is with respect to any independent variable say x_i is computed by treating the remaining independent variables as constant.

Examples

1. Prove that $y v_y - x v_x = -y^2 v^3$ if $v = (1 - 2xy + y^2)^{-1/2}$.
2. Show that $w_x + w_y + w_z = 0$ if $w = (y - z)(z - x)(x - y)$.
3. Show that $u_x + u_y = u$, if $u = \frac{e^{x+y}}{e^x + e^y}$.
4. If $w = x^2 y + y^2 z + z^2 x$, prove that $w_x + w_y + w_z = (x + y + z)^2$.
5. If $u = \frac{y}{z} + \frac{z}{x}$, show that $xu_x + yu_y + zu_z = 0$.
6. Given $u = e^{r \cos \theta} \cos(r \sin \theta)$, $v = e^{r \cos \theta} \sin(r \sin \theta)$, Prove that $\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$.
7. If $(x + y)z = x^2 + y^2$, Show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)^2 = 4 \left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right]$.
8. Show that $(v_x)^2 + (v_y)^2 + (v_z)^2 = v^4$ if $v = (x^2 + y^2 + z^2)^{-1/2}$.
9. If $z = e^{ax+by} f(ax - by)$ prove that $b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2abz$.
10. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, then show that (i) $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$ and (ii) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.
11. If $u = \log_e(x^3 + y^3 - x^2 y - xy^2)$, then show that $u_{xx} + 2u_{xy} + u_{yy} = -\frac{4}{(x + y)^2}$.
12. If $z = f(x + ct) + \phi(x - ct)$, then prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$.
13. If $u = e^{a\theta} \cos(a \log_e r)$ Show that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$.
14. Prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, if (a) $u = x^y$, (b) $u = \log_e \left(\frac{x^2 + y^2}{xy} \right)$, (c) $u = \sin^{-1}(y/x)$
15. Prove that $u_{xx} + u_{yy} = 0$, if (a) $u = \tan^{-1} \left(\frac{2xy}{x^2 - y^2} \right)$, (b) $u = \log_e(x^2 + y^2) + \tan^{-1}(y/x)$
16. Find the value of n so that the equation $v = r^n (3 \cos^2 \theta - 1)$ satisfies the relation $\frac{\partial}{\partial r} \left(r^2 \frac{\partial v}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v}{\partial \theta} \right) = 0$. **Ans: $n = -3, 2$**



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17. If $Z = \tan(y + ax) + (y - ax)^{3/2}$, Show that $Z_{xx} = a^2 Z_{yy}$.
18. If $v = \frac{1}{\sqrt{t}} e^{-x^2/4a^2t}$, Prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$.
19. If $\theta = t^n e^{-\left(\frac{r^2}{4t}\right)}$, what value of n will make $\frac{1}{r^2} \left[\frac{\partial}{\partial r} \left(r^2 \frac{\partial \theta}{\partial r} \right) \right] = \frac{\partial \theta}{\partial t}$. **Ans: $n = -3/2$**
20. If $x^x y^y z^z = c$, show that $z_{xy} = -[x \log_e(e^x)]^{-1}$, when $x = y = z$.
21. If $z = \log_e(e^x + e^y)$, Show that $rt - s^2 = 0$ where $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$.
22. If $v = \log_e(x^2 + y^2 + z^2)$, prove that $(x^2 + y^2 + z^2)[v_{xx} + v_{yy} + v_{zz}] = 2$.
23. If $u = \log_e[x^3 + y^3 + z^3 - 3xyz]$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = \frac{-9}{(x + y + z)^2}$.
24. If $w = r^m$, prove that $w_{xx} + w_{yy} + w_{zz} = m(m+1)r^{m-2}$ where $r^2 = x^2 + y^2 + z^2$.
25. Verify that v satisfies Laplace's equation $v_{xx} + v_{yy} + v_{zz} = 0$ if
- a) $v = e^{3x+4y} \cos 5z$ b) $v = (x^2 + y^2 + z^2)^{-1/2}$ c) $v = \cos 3x \cos 4y \sinh 5z$

I. COMPOSITE FUNCTIONS

Total Derivative:

1) If $u = f(x, y)$ where $x = \phi(t)$ and $y = \psi(t)$ then $\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$.

Examples

1. Find the total differential of the following functions:

a) $f(x, y) = x \cos y - y \cos x$ **Ans:** $df(x, y) = (\cos(y) + y \sin(x))dx + (-\sin(y) - \cos(x))dy$

(b) $f(x, y) = e^{xyz}$ **Ans:** $df(x, y, z) = e^{xyz} yz dx + e^{xyz} xz dy + e^{xyz} xy dz$

2. Find $\frac{du}{dt}$ for the following functions:

a) $u = x^2 - y^2$, $x = e^t \cos t$, $y = e^t \sin t$ at $t = 0$. **Ans:** $\frac{\partial u}{\partial t} = 0$ at $t = 0$

b) $u = x^y$, when $y = \tan^{-1} t$, $x = \sin t$

Ans: $\frac{\partial u}{\partial t} = \tan^{-1}(t) \cdot \sin(t)^{\tan^{-1}(t)-1} \cos(t) + \log(\sin(t)) \cdot \sin(t)^{\tan^{-1}(t)} \cdot \frac{1}{1+t^2}$

c) $u = \sin(e^x + y)$, $x = f(t)$, $y = g(t)$



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Ans: $\frac{\partial u}{\partial t} = \cos[e^{f(t)} + g(t)] \cdot e^{f(t)} f'(t) + \cos[e^{f(t)} + g(t)] \cdot g'(t)$

d) $u = \tan^{-1}(y/x), x = e^t - e^{-t}, y = e^t + e^{-t}$

Ans: $\frac{\partial u}{\partial t} = \frac{x^2 - y^2}{x^2 + y^2}$

e) $u = xy + yz + zx, x = \frac{1}{t}, y = e^t$ and $z = e^{-t}$

Ans: $\frac{\partial u}{\partial t} = \frac{xt^2(y-z) - (y+z)}{t^2}$

f) $u = x^3 ye^z$ where $x = t, y = t^2$ and $z = \log_e t$ at $t = 2$.

Ans: $\frac{\partial u}{\partial t} = 192$ at $t = 2$

g) $u = \sin\left(\frac{x}{y}\right)$ and $x = e^t, y = t^2$.

Ans: $\frac{\partial u}{\partial t} = \frac{e^t \cos\left(\frac{e^t}{t^2}\right)(t-2)}{t^3}$

h) $u = x^2 + y^2 + z^2, x = e^{2t}, y = e^{2t} \cos 3t$ and $z = e^{2t} \sin 3t$ **Ans:** $\frac{\partial u}{\partial t} = 4u$ or $4(x^2 + y^2 + z^2)$

2. The altitude of a right circular cone is 15cm and is increasing at 0.2cm/s. The radius of the base is 10cm and is decreasing at 0.3cm/s. How fast is the volume changing?

Ans: decreases by 73.267cm³/sec.

3. Find the rate at which the area of a rectangle is increasing at a given instant when the sides of the rectangle are 4 ft and 3 ft and are increasing at the rate of 1.5 fts⁻¹ and 0.5 fts⁻¹ respectively.

Ans: $\frac{\partial A}{\partial t} = 6.5 \text{ ft}^2/\text{sec}.$

4. In order that the function $u = 2xy - 3x^2y$ remains constant, what should be the rate of change of y w.r.t t, given x increases at the rate of 2cm/sec at the instant when $x = 3\text{cm}$ and $y = 1\text{cm}$.

Ans: $\frac{dy}{dt} = -32 \text{ cm/sec}.$

5. The temperature function for a bird in flight is given by $T(x, y, z) = 0.9x^2 + 1.4xy + 95z^2$. Find approximately the change in temperature when head wind x increases from 1 meter per second to 2 meters per second, bird heart rate y increases from 50 beats per minute to 55 beats per minute and flapping rate z increases from 3 flaps per second to 4 flaps per second.

Hint: $dT = \frac{dT}{dx} \cdot dx + \frac{dT}{dy} \cdot dy + \frac{dT}{dz} \cdot dz$

Ans: dT = 573.08 units

6. Elevation of land above sea level, H, depends on two map coordinates (x, y) in the following way:

$H(x, y) = e^{-0.01(x^2 + y^2)}$. A car travels thorough this terrain, so its coordinates depend on time in the



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following way: $x(t) = -7 + 10\cos(10t)$, $y(t) = 4 + 10\sin(10t)$. Find the speed with which the altitude of the car increases or decreases at $t = 0$.

Ans: $\frac{dH}{dt} = -6.2304 \text{ units.}$

7. If $w = xy + z$ is the temperature at any point (x, y, z) along a curve C with parametric equations $x = \cos t$, $y = \sin t$, $z = t$. Find the instantaneous rate of change of temperature due to the motion along the curve.

Ans: $\frac{dw}{dt} = \cos(2t) + 1.$

8. The voltage V in a circuit that satisfies the law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Find how the current is changing at the instant when $R = 600 \text{ ohms}$, $I = 0.04 \text{ amp}$, $\frac{dR}{dt} = 0.5 \text{ ohm/sec}$, and $\frac{dV}{dt} = -0.01 \text{ volt/sec}$

Ans: $\frac{dI}{dt} = -0.0001 \text{ amp/sec.}$

Partial differentiation of Composite functions:

1) If $u = f(x, y)$ where $x = g(s, t)$ and $y = h(s, t)$ then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$$

2) If $u = f(x, y, z)$ where $x = g(r, s, t)$, $y = h(r, s, t)$ and $z = k(r, s, t)$ then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial r}, \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s} \quad \text{and}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t}$$

Examples

1. If $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$ find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$.

Ans: $\frac{\partial u}{\partial r} = 4x + 2y$, $\frac{\partial u}{\partial s} = -6x - 16y$

2. If $W = u^2v$ and $u = e^{x^2+y^2}$, $v = \sin(xy^2)$ find $\frac{\partial W}{\partial x}$ and $\frac{\partial W}{\partial y}$.

Ans: $\frac{\partial W}{\partial x} = (e^{x^2+y^2})^2 [4x \sin(xy^2) + y^2 \cos(xy^2)]$, $\frac{\partial W}{\partial y} = (e^{x^2+y^2})^2 [4y \sin(xy^2) + 2x \cos(xy^2)]$

3. If $u = u\left[\frac{y-x}{xy}, \frac{z-x}{xz}\right]$, show that $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$.

4. If $u = F(x-y, y-z, z-x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.



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5. If $u = F(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2} \frac{\partial u}{\partial x} + \frac{1}{3} \frac{\partial u}{\partial y} + \frac{1}{4} \frac{\partial u}{\partial z} = 0$.

6. If $u = F(e^{y-z}, e^{z-x}, e^{x-y})$, Find the value of $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

Ans: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

7. Prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$ if z is a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$.

8. If $V = f(r, s, t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$ show that $x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} = 0$.

9. If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and F is a function of x, y, z show that

$$u \frac{\partial F}{\partial u} + v \frac{\partial F}{\partial v} + w \frac{\partial F}{\partial w} = x \frac{\partial F}{\partial x} + 2y \frac{\partial F}{\partial y} + 3z \frac{\partial F}{\partial z}$$

10. If z is a function of x and y and $x = e^u \cos v$, $y = e^u \sin v$.

Prove that (i) $x \frac{\partial z}{\partial v} + y \frac{\partial z}{\partial u} = e^{2u} \frac{\partial z}{\partial y}$

$$(ii) \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 = e^{-2u} \left[\left(\frac{\partial z}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2 \right]$$

II. JACOBIAN

If u and v are functions of two independent variable x and y then the Jacobian of u, v w.r.t

x, y is denoted by $\frac{\partial(u, v)}{\partial(x, y)}$ or $J \left(\frac{u, v}{x, y} \right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

Properties of Jacobian

1. If u, v are functions of x, y and x, y are functions of u, v .

$$J = \frac{\partial(u, v)}{\partial(x, y)} \quad \text{and} \quad J' = \frac{\partial(x, y)}{\partial(u, v)} \quad \text{then} \quad JJ' = 1$$

2. If u, v are functions of r, s and r, s are functions of x, y then $\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \times \frac{\partial(r, s)}{\partial(x, y)}$.

3. Two functions u and v are functionally dependent if their Jacobian $J \left(\frac{u, v}{x, y} \right) = 0$

Examples

1. If $u = a \cosh x \cos y$, $v = a \sinh x \sin y$ Show that $\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{2} a^2 (\cosh 2x - \cos 2y)$.



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2. Find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ for the following:

a) $u = x^2, v = \sin y, w = e^{-3z}$.

Ans: $\frac{\partial(u, v, w)}{\partial(x, y, z)} = -6xe^{-3z}\cos(y)$

b) $u = x^2 + y^2 + z^2, v = xy + yz + zx, w = x + y + z$.

Ans: $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$.

c) $u = x + 3y^2 - z^3, v = 4x^2yz, w = 2z^2 - xy$ at $(1, -1, 0)$.

Ans: $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 20$ at $(1, -1, 0)$.

d) $u = \frac{2yz}{x}, v = \frac{3zx}{y}, w = \frac{4xy}{z}$

Ans: $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 96$

3. Find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$ if

a) $x = \frac{u^2 - v^2}{2}, y = uv, z = w$.

Ans: $\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 + v^2$.

b) $u = x + y + z, uv = y + z, uvw = z$,

Ans: $\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2v$.

4. Verify that $JJ' = 1$

a) $x = e^u \cos v, y = e^u \sin v$.

b) $x = u(1 - v), y = uv$.

c) $u = x + \frac{y^2}{x}, v = \frac{y^2}{x}$.

d) $x = u, y = u \tan v, z = w$

e) $y_1 = \frac{x_2 x_3}{x_1}, y_2 = \frac{x_3 x_1}{x_2}, y_3 = \frac{x_1 x_2}{x_3}$

f) If $x = e^u \sec(v), y = e^u \tan(v)$

5. Find $\frac{\partial(u, v)}{\partial(r, \theta)}$ when,

a) $u = x^2 - 2y^2, v = 2x^2 - y^2$ and $x = r \cos \theta, y = r \sin \theta$

Ans: $\frac{\partial(u, v)}{\partial(r, \theta)} = \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, \theta)} = 6r^3 \sin(2\theta)$

b) $u = x^2 - y^2, v = 2xy$ and $x = r \cos \theta, y = r \sin \theta$.

Ans: $\frac{\partial(u, v)}{\partial(r, \theta)} = 4r^3$

c) $u = 2xy, v = x^2 - y^2$ and $x = r \cos \theta, y = r \sin \theta$

Ans: $\frac{\partial(u, v)}{\partial(r, \theta)} = -4r^3$

d) $u = 2axy$ & $v = a(x^2 - y^2)$ and $x = r \cos \theta, y = r \sin \theta$

Ans: $\frac{\partial(u, v)}{\partial(r, \theta)} = -4a^2r^3$

6. Find $\frac{\partial(X, Y)}{\partial(x, y)}$ where $X = u^2v, Y = uv^2$ and $u = x^2 - y^2, v = yx$

Ans: $\frac{\partial(X, Y)}{\partial(x, y)} = 6x^2y^2(x^2 - y^2)(x^4 - y^4)$

7. If $x = \sqrt{vw}, y = \sqrt{wu}, z = \sqrt{uv}$ and $u = r \sin \theta \cos \phi, v = r \sin \theta \sin \phi, w = r \cos \theta$, calculate



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$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}.$$

$$\text{Ans: } \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \frac{\partial(x, y, z)}{\partial(u, v, w)} \cdot \frac{\partial(u, v, w)}{\partial(r, \theta, \phi)} = \frac{r^2 \sin(\theta)}{4}$$

8. Show that the following u & v are functionally dependent and hence find their functional relationship. (**Hint:** If $\frac{\partial(u, v)}{\partial(x, y)} = 0$ then, u and v are functionally dependent.)

(i) $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $v = \sin^{-1} x + \sin^{-1} y$ **Ans:** $v = \sin^{-1} u$

(ii) $u = \frac{x+y}{1-xy}$, $v = \tan^{-1} x + \tan^{-1} y$ **Ans:** $v = \tan^{-1} u$

(iii) $u = e^x \sin y$, $v = x + \log_e(\sin y)$. **Ans:** $v = \log_e u$

III. TAYLOR'S SERIES FOR FUNCTION OF TWO VARIABLES

$$\begin{aligned} f(x, y) = & f(a, b) + \left[(x-a) \frac{\partial f(a, b)}{\partial x} + (y-b) \frac{\partial f(a, b)}{\partial y} \right] \\ & + \frac{1}{2!} \left[(x-a)^2 \frac{\partial^2 f(a, b)}{\partial x^2} + (y-b)^2 \frac{\partial^2 f(a, b)}{\partial y^2} + 2(x-a)(y-b) \frac{\partial^2 f(a, b)}{\partial x \partial y} \right] \\ & + \frac{1}{3!} \left[(x-a)^3 \frac{\partial^3 f(a, b)}{\partial x^3} + (y-b)^3 \frac{\partial^3 f(a, b)}{\partial y^3} + 3(x-a)^2(y-b) \frac{\partial^3 f(a, b)}{\partial x^2 \partial y} \right. \\ & \left. + 3(x-a)(y-b)^2 \frac{\partial^3 f(a, b)}{\partial x \partial y^2} \right] + \dots \end{aligned}$$

Examples

Expand the following functions up to second-degree terms.

1. $f(x, y) = x^2 + xy + y^2$ in powers of $(x-1)$ and $(y-2)$.

Ans: $f(x, y) = 7 + 4(x-1) + 5(y-2) + (x-1)^2 + (y-2)^2 + (x-1)(y-2) \dots \dots \dots$

2. $f(x, y) = (1+x+y)^{-1}$ in powers of $(x-1)$ and $(y-1)$.

Ans: $f(x, y) = \frac{1}{3} - \frac{1}{3}[(x-1) + (y-1)] + \frac{1}{3^2}[(x-1)^2 + (y-1)^2 + 2(x-1)(y-1)] \dots \dots$

3. $f(x, y) = e^x \cos y$ in powers of $(x-1)$ and $(y-\pi/4)$.

Ans: $f(x, y) = \frac{e}{\sqrt{2}} \left\{ 1 + (x-1) - \left(y - \frac{\pi}{4}\right) + \frac{1}{2} \left[(x-1)^2 - \left(y - \frac{\pi}{4}\right)^2 - 2(x-1) \left(y - \frac{\pi}{4}\right) \right] \right\}$

4. $f(x, y) = x^y$ in powers of $(x-1)$ and $(y-1)$ and also find $(1.1)^{1.1}$.



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Ans: $f(x, y) = 1 + (x - 1) + (x - 1)(y - 1) \dots = 1.1111$

5. $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ in powers of $(x - 1)$ and $(y - 1)$.

Ans: $f(x, y) = \frac{\pi}{4} - \frac{1}{2}\{(x - 1) - (y - 1)\} + \frac{1}{4}\{(x - 1)^2 - (y - 1)^2 \dots\}$

6. $f(x, y) = x^3 + xy^2 + y^3$ in powers of $(x - 1)$ and $(y - 2)$.

Ans: $f(x, y) = 13 + 7(x - 1) + 16(y - 2) + 3(x - 1)^2 + 7(y - 2)^2 + 4(x - 1)(y - 2) \dots$

7. $f(x, y) = \cot^{-1}(xy)$ in powers of $(x + 0.5)$ and $(y - 2)$ and hence compute $f(-0.4, 2.2)$.

Ans: $f(x, y) = -\frac{\pi}{4} - 0.1625$

8. $f(x, y) = x^2y + 3y - 2$ in powers of $(x + 1)$ and $(y - 2)$.

Ans: $f(x, y) = 6 - 4\{(x + 1) - (y - 2)\} + 2(x + 1)^2 - 2(x + 1)(y - 2) \dots$

9. $f(x, y) = \sin(xy)$ in powers of $(x - 1)$ and $(y - \frac{\pi}{2})$

Ans: $f(x, y) = 1 - \frac{\pi^2}{8}(x - 1)^2 - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 - \frac{\pi}{2}(x - 1)\left(y - \frac{\pi}{4}\right) \dots$

10. $f(x, y) = xy^2 + \cos xy$ in powers of $(x - 1)$ and $(y - \frac{\pi}{2})$.

Ans: $f(x, y) = \frac{\pi^2}{4} + \left(\frac{\pi^2}{4} - \frac{\pi}{2}\right)(x - 1) + (\pi - 1)\left(y - \frac{\pi}{2}\right) + \left(y - \frac{\pi}{2}\right)^2 + \pi(x - 1)\left(y - \frac{\pi}{2}\right) \dots$

11. $f(x, y) = x^2y + \sin y + e^x$ about $(1, \pi)$.

Ans: $f(x, y) = \pi + e + (2\pi + e)\left[(x - 1) + \frac{(x - 1)^2}{2}\right] + 2(x - 1)(y - \pi) \dots$

12. $f(x, y) = e^x \sin y$ about $(-1, \pi/4)$.

Ans: $f(x, y) = \frac{1}{e\sqrt{2}}\left\{1 + (x + 1) + \left(y - \frac{\pi}{4}\right) + \frac{1}{2}\left[(x + 1)^2 - \left(y - \frac{\pi}{4}\right)^2 + 2(x + 1)\left(y - \frac{\pi}{4}\right)\right]\right\}$

IV. MACLAURIN'S SERIES FOR FUNCTION OF TWO VARIABLES



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$$f(x, y) = f(0, 0) + \left[x \frac{\partial f(0, 0)}{\partial x} + y \frac{\partial f(0, 0)}{\partial y} \right] + \frac{1}{2!} \left[x^2 \frac{\partial^2 f(0, 0)}{\partial x^2} + y^2 \frac{\partial^2 f(0, 0)}{\partial y^2} + 2xy \frac{\partial^2 f(0, 0)}{\partial x \partial y} \right] + \frac{1}{3!} \left[x^3 \frac{\partial^3 f(0, 0)}{\partial x^3} + y^3 \frac{\partial^3 f(0, 0)}{\partial y^3} + 3x^2 y \frac{\partial^3 f(0, 0)}{\partial x^2 \partial y} + 3xy^2 \frac{\partial^3 f(0, 0)}{\partial x \partial y^2} \right] + \dots$$

Examples

1. $\cos x \cos y$ in powers of x and y up to second-degree terms.

Ans: $f(x, y) = 1 - \frac{x^2 + y^2}{2}$

2. Expand $e^y \log_e(1+x)$ in powers of x and y up to third-degree terms.

Ans: $f(x, y) = x - \frac{x^2}{2} + xy + \frac{x^3}{3} - \frac{x^2 y}{2} + \frac{xy^2}{2} \dots$

3. Expand $e^{ax} \sin by$ in the ascending powers of x and y up to third-degree terms.

Ans: $f(x, y) = by + abxy + \frac{a^2 bx^2 y}{2} - \frac{b^3 y^3}{6} \dots \dots$

4. Find the Maclaurin's expansion of $e^x \log_e(1+y)$ up to third-degree terms.

Ans: $f(x, y) = y - \frac{y^2}{2} + xy + \frac{y^3}{3} + \frac{x^2 y}{2} - \frac{xy^2}{2} \dots \dots$

5. Expand $\log_e(1+x+y)$ in the neighbourhood of $(0,0)$ up to second degree terms.

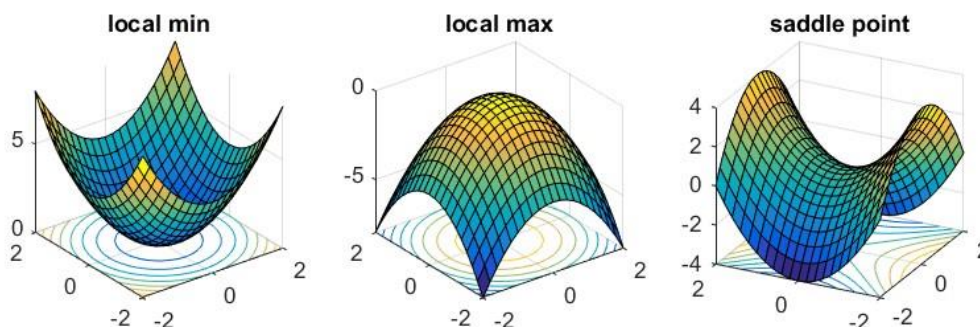
Ans: $f(x, y) = (x + y) - \frac{1}{2}(x + y)^2 \dots \dots \dots$

V. MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

- A function $z = f(x, y)$ is said to have a maximum or minimum at $x = a, y = b$ as $f(a+h, b+k) < f(a, b)$ or $f(a+h, b+k) > f(a, b)$ for all values of h and k .
- Necessary conditions for $f(x, y)$ to have a maximum or a minimum at (a, b) are $f_x(a, b) = 0$ and $f_y(a, b) = 0$.
- Given the function $z = f(x, y)$, the point $(x_0, y_0, f(x_0, y_0))$ is a saddle point if $f_x(x_0, y_0) = 0$, $f_y(x_0, y_0) = 0$ but f does not have a local extremum at (x_0, y_0) .



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Working rule to find the maxima and minima of $z = f(x, y)$

- 1) Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ and equate them to zero, solve these as simultaneous equations in x and y .
 Let $(a, b), (c, d), \dots$ be the roots of the simultaneous equations.
- 2) Calculate the values of $r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$ for each of points.
- 3) If $rt - s^2 > 0$ and $r < 0$ at (a, b) , $f(x, y)$ has a maximum value.
- 4) If $rt - s^2 > 0$ and $r > 0$ at (a, b) , $f(x, y)$ has a minimum value.
- 5) If $rt - s^2 < 0$ at (a, b) , (a, b) is a saddle point.
- 6) If $rt - s^2 = 0$ at (a, b) then the case is doubtful and needs further investigation.

Examples

1. Find the extreme values of the following functions:

a) $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Ans: Maximum Point $(4, 0)$,

$f(4, 0) = 120$ and Minimum Point $(6, 0)$, $f(6, 0) = 108$.

b) $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$.

Ans: Max Point $(1, 0)$ & $(-1, 0)$, $f(1, 0) = f(-1, 0) = 1$

Min Point $(0, 1)$ & $(0, -1)$, $f(0, 1) = f(0, -1) = -1$.

c) $f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$

Ans: Minimum Point at (a, a) and $f(a, a) = 3a^2$.

d) $f(x, y) = x^4 + y^4 - x^2 - y^2 + 1$

Ans: Maximum Point $(0, 0)$ then $f(0, 0) = 1$ and Minimum value, $f(a, b) = \frac{1}{2}$.



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e) $f(x, y) = x^2 + 2y^2 + 3z^2 - 2xy - 2yz - 2$

Ans: Minimum functional value $f(a, b) = 2z^2 - 2$.

f) $f(x, y) = x^3y^2(1-x-y)$

Ans: Maximum Point $\left(\frac{1}{2}, \frac{1}{3}\right)$, then $f\left(\frac{1}{2}, \frac{1}{3}\right) = \frac{1}{432}$.

g) $f(x, y) = x^3 + y^3 - 3y - 12x + 20$

Ans: Max Point $(-2, -1)$ then $f(-2, -1) = 38$ and Mini value $(2, 1)$ then $f(2, 1) = 2$.

h) $f(x, y) = \cos x \cos y \cos(x+y)$

Ans: Minimum Point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ then $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = -\frac{1}{8}$

i) $x^2 + y^2 + 6x + 12 = 0$

Ans: Minimum Point $(-3, 0)$ then $f(-3, 0) = -3$

j) $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Ans: Minimum Point $(\pm\sqrt{2}, \pm\sqrt{2})$ then $f(\pm\sqrt{2}, \pm\sqrt{2}) = -8$

2. Find the shortest distance from origin to the plane $x - 2y - 2z = 3$

Ans: Minimum Point $\left(\frac{1}{3}, \frac{-2}{3}\right)$ then $f\left(\frac{1}{3}, \frac{-2}{3}\right) = 1$.

3. Find the points on the surface $z^2 = xy + 1$ nearest to the origin.

Ans: Minimum Point or the point near to the origin is $(0, 0, 1)$.

4. Find the shortest distance from origin to the surface $xyz^2 = 2$.

Ans: Minimum Point $(\pm 1, \pm 1)$ and the shortest distance, $d = 2$ units.

5. Sum of three numbers is a constant. Prove that their product is maximum when they are equal.

Ans: The three numbers are $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$ and Product is maximum, value is $\frac{a^3}{27}$.

6. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.

Ans: The value of 3 parts is $(x, y, z) = (40, 40, 40)$

7. Examine the function $f(x, y) = \sin x + \sin y + \sin(x+y)$, $x, y \in (0, \pi)$ for extreme values.

Ans: Maximum Point $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ then functional value is $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$.

8. In a plane triangle find the maximum value of $\cos A \cos B \cos C$ where A, B and C are the angles of the triangle.

Ans: Maximum Point $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$ then functional value is $f\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{1}{8}$.



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9. Find the minimum value of $x^2 + y^2 + z^2$ given $ax + by + cz = p$

Ans: The minimum functional value is $= \frac{p^2}{a^2 + b^2 + c^2}$.

10. A flat circular plate is heated so that the temperature at any point (x, y) is $u(x, y) = x^2 + 2y^2 - x$. Find the coldest point on the plate.

Ans: The minimum functional value at $(\frac{1}{2}, 0)$ is $f(\frac{1}{2}, 0) = -0.25$ or $-\frac{1}{4}$.

11. A rectangular box open at the top is to have a volume 108 cubic meters. Find its dimensions if its total surface area is minimum.

Ans: The Dimensions for the minimum Surface area is length = breadth = 6m, height = 3m.

12. The temperature T at any point (x, y, z) in space is $T(x, y, z) = kxyz^2$ where k is a positive constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Ans: The highest temperature value is $= \frac{ka^4}{8}$.

13. Assume that you are in charge of erecting a radio telescope on a newly discovered planet. To minimize interference, you want to place it where the magnetic field of the planet is weakest. The planet is spherical with a radius of 6 units. Based on a coordinate system whose origin is at the center of the planet, the strength of the magnetic field is given by $M(x, y, z) = 6x - y^2 + xz + 60$. Where should you locate the radio telescope?

Ans: The radio telescope should be located at either of the points $(-4, 4, 2)$ or $(-4, -4, 2)$.