BMS COLLEGE OF ENGINEERING, BENGALURU-19
Autonomous Institute, Affiliated to VTU

DEPARTMENT OF MATHEMATICS

For the Course Code: 23MA1BSMCS

I. PARTIAL DERIVATIVES

Let $u = f(x, y)$ be a function of two variables x and y. If we keep y as constant and vary x alone, then u is a function of x only. The derivative of u with respect to x, treating y as constant is called the partial derivative of u w. r. t x and is denoted by one of the symbols $\frac{\partial u}{\partial x}$, u_x, Thus $\frac{\partial u}{\partial x} = u_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$. Similarly $\frac{\partial u}{\partial y} = u_y = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$

Further u_x and u_y are also functions of x and y, so these can further be differentiated partially w. r. t. x and y. Thus $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$ or u_{xx}, $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$ or u_{xy}, $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$ or u_{yx} and $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$ or u_{yy}.

Examples

1. Prove that $y v_y - x v_x = -y^2 v^3$ if $v = \left(1 - 2xy + y^2\right)^{-1/2}$.
2. Show that $w_x + w_y + w_z = 0$ *if* $w = (y-z)(z-x)(x-y)$.
3. Show that $u_x + u_y = u$, if $u = \frac{e^{x+y}}{e^x + e^y}$.
4. If $w = x^2 y + y^2 z + z^2 x$, prove that $w_x + w_y + w_z = (x+y+z)^2$.
5. If $u = \frac{y}{z} + \frac{z}{x}$, show that $xu_x + yu_y + zu_z = 0$.
6. Given $u = e^{r\cos\theta}\cos(r\sin\theta)$, $v = e^{r\cos\theta}\sin(r\sin\theta)$, Prove that

   $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$.
7. If $(x+y)z = x^2 + y^2$, Show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$.
8. Show that $(v_x)^2 + (v_y)^2 + (v_z)^2 = v^4$ if $v = (x^2 + y^2 + z^2)^{-1/2}$.

9. If $z = e^{ax+by} f(ax-by)$ prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

10. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, then show that

(i) $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$ and (ii) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

11. If $u = \log_e\left(x^3 + y^3 - x^2y - xy^2\right)$, then show that $u_{xx} + 2u_{xy} + u_{yy} = -\frac{4}{(x+y)^2}$.

12. If $z = f(x+ct) + \phi(x-ct)$, then prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$.

13. If $u = e^{a\theta}\cos(a\log_e r)$ Show that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$.

14. Prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, if

a) $u = x^y$

b) $u = \log_e\left(\frac{x^2 + y^2}{xy}\right)$

15. Verify that $f_{xy} = f_{yx}$ when $f(x,y) = \sin^{-1}(y/x)$.

16. Prove that $u_{xx} + u_{yy} = 0$, if

a) $u = \tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)$

b) $u = \log_e\left(x^2 + y^2\right) + \tan^{-1}(y/x)$

17. Find the value of n so that the equation $v = r^n\left(3\cos^2\theta - 1\right)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) = 0.$$

18. If $Z = \tan(y+ax) + (y-ax)^{3/2}$, Show that $Z_{xx} = a^2 Z_{yy}$.

19. If $v = \frac{1}{\sqrt{t}} e^{-x^2/4a^2t}$, Prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$.

20. If $\theta = t^n e^{-r^2/4t}$, what value of n will make $\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)\right] = \frac{\partial\theta}{\partial t}$.

21. If $x^x y^y z^z = c$, show that $z_{xy} = -\left[x\log_e(ex)\right]^{-1}$, when $x = y = z$.

22. If $z = \log_e\left(e^x + e^y\right)$, Show that $rt - s^2 = 0$ where $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$.

23. If $v = \log_e(x^2 + y^2 + z^2)$, prove that $(x^2 + y^2 + z^2)\left[v_{xx} + v_{yy} + v_{zz}\right] = 2$.

24. If $u = \log_e\left[x^3 + y^3 + z^3 - 3xyz\right]$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

25. If $w = r^m$, prove that $w_{xx} + w_{yy} + w_{zz} = m(m+1)r^{m-2}$ where $r^2 = x^2 + y^2 + z^2$.

26. Verify that v satisfies Laplace's equation $v_{xx} + v_{yy} + v_{zz} = 0$ if

a) $v = e^{3x+4y}\cos 5z$

b) $v = (x^2 + y^2 + z^2)^{-1/2}$

c) $v = \cos 3x \cos 4y \sinh 5z$

II. COMPOSITE FUNCTIONS

Total Derivative:

If $u = f(x, y)$ where $x = \phi(t)$ and $y = \psi(t)$ then $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$.

Examples

1. Find the differential of the following functions:
   a) $f(x, y) = x\cos y - y\cos x$
2. $f(x, y) = e^{xyz}$
3. Find $\frac{du}{dt}$ for the following functions:
   a) $u = x^2 - y^2,\ x = e^t \cos t,\ y = e^t \sin t$ at $t = 0$.
   b) $u = x^y$, when $y = \tan^{-1} t$, $x = \sin t$
   c) $u = \sin(e^x + y)$, $x = f(t)$, $y = g(t)$
   d) $u = \tan^{-1}(y/x),\ x = e^t - e^{-t},\ y = e^t + e^{-t}$
   e) $u = xy + yz + zx, x = 1/t,\ y = e^t$ and $z = e^{-t}$
   f) $u = x^3 y e^z$ where x = t, y = t^2 and $z = \log_e t$, at t = 2.
4. Find $\frac{du}{dt}$ and verify the result by direct substitution.
   a) $u = \sin\left(\frac{x}{y}\right)$ and $x = e^t, y = t^2$.
   b) $u = x^2 + y^2 + z^2,\ x = e^{2t},\ y = e^{2t}\cos 3t$ and $z = e^{2t}\sin 3t$
5. If $u = \sin^{-1}(x - y),\ x = 3t,\ y = 4t^3$, show that $\frac{du}{dt} = 3(1 - t^2)^{-\frac{1}{2}}$. Also verify the result by direct substitution.
6. The altitude of a right circular cone is $15cm$ and is increasing at $0.2\,cm/s$. The radius of the base is $10cm$ and is decreasing at $0.3\,cm/s$. How fast is the volume changing?

7. Find the rate at which the area of a rectangle is increasing at a given instant when the sides of the rectangle are 4 ft and 3 ft and are increasing at the rate of 1.5 fts^{-1} and 0.5 fts^{-1} respectively.
8. In order that the function $u = 2xy - 3x^2 y$ remains constant, what should be the rate of change of y w.r.t. t, given x increases at the rate of 2cm/sec at the instant when x = 3cm and y = 1cm.

Partial differentiation of Composite functions:

If $u = f(x, y)$ where $x = g(s,t)$ and $y = h(s,t)$ then

$$\frac{\partial u}{ds} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial u}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

Examples

1. If $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$ find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$.
2. If $W = u^2 v$ and $u = e^{x^2 - y^2}$, $v = \sin(xy^2)$ find $\frac{\partial W}{\partial x}$ and $\frac{\partial W}{\partial y}$.
3. If $u = u\left[\frac{y-x}{xy}, \frac{z-x}{xz}\right]$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.
4. If $u = F(x-y, y-z, z-x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.
5. Prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$ if z is a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$.
6. If $V = f(r,s,t)$ and $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$ show that $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = 0$.
7. If $x = u + v + w, y = vw + wu + uv, z = uvw$ and F is a function of x, y, z show that
$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$$
8. If z is a function of x and y and $x = e^u \cos v$, $y = e^u \sin v$.

   Prove that (i) $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$ (ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

III. JACOBIAN

If u and v are functions of two independent variable x and y then the Jacobian of u , v w.r.to x , y is denoted by

$$.\frac{\partial(u,v)}{\partial(x,y)} \text{ or } J\left(\frac{u,v}{x,y}\right)=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Properties of Jacobians

1. If u,v are functions of x,y and x,y are functions of u,v
   $J=\dfrac{\partial(u,v)}{\partial(x,y)}$ and $J'=\dfrac{\partial(x,y)}{\partial(u,v)}$ then $JJ'=1$.
2. If u,v are functions of r,s and r,s are functions of x,y then

$$\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}\times\frac{\partial(r,s)}{\partial(x,y)}.$$

Examples

1. Find $\dfrac{\partial(u,v)}{\partial(x,y)}$ for the following:
   a) $u=e^x\sin y,\ v=x+\log_e(\sin y)$.
   b) $u=x+y,\ \ y=uv$
2. If $x=a\cosh\xi\cos\eta,\ y=a\sinh\xi\sin\eta$ Show that $\dfrac{\partial(x,y)}{\partial(\xi,\eta)}=\dfrac{1}{2}a^2(\cosh 2\xi-\cos 2\eta)$.
3. Find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ for the following:
   a) $u=x^2,\ v=\sin y,\ w=e^{-3z}$.
   b) $u=x^2+y^2+z^2,\ v=xy+yz+zx,\ w=x+y+z$.
   c) $u=x+3y^2-z^3, v=4x^2yz,\ w=2z^2-xy$ at (1,-1,0).
   d) $u=\dfrac{2yz}{x},\ v=\dfrac{3zx}{y},\ w=\dfrac{4xy}{z}$
4. If $y_1=\dfrac{x_2x_3}{x_1},\ y_2=\dfrac{x_3x_1}{x_2},\ y_3=\dfrac{x_1x_2}{x_3}$, then show that $\dfrac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}=4$.
5. Find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ if
   a) $x=\dfrac{u^2-v^2}{2}, y=uv,\ z=w$.
   b) $u=x+y+z,\ uv=y+z,\ uvw=z$,
6. Verify that $JJ'=1$
   a) $x=e^u\cos v,\ y=e^u\sin v$.

b) $x = u(1-v),\ y = uv$.

c) $u = x + \frac{y^2}{x},\ v = \frac{y^2}{x}$.

d) $x = u, y = u\tan v,\ z = w$

7. Find $\frac{\partial(u,v)}{\partial(r,\theta)}$ when,

a) $u = x^2 - 2y^2, v = 2x^2 - y^2$ and $x = r\cos\theta,\ y = r\sin\theta$

b) $u = x^2 - y^2, v = 2xy$ and $x = r\cos\theta,\ y = r\sin\theta$

c) $u = 2xy, v = x^2 - y^2$ and $x = r\cos\theta,\ y = r\sin\theta$

d) $u = 2axy$ & $v = a\left(x^2 - y^2\right)$ and $x = r\cos\theta,\ y = r\sin\theta$

8. Find $\frac{\partial(X,Y)}{\partial(x,y)}$ where $X = u^2 v, Y = uv^2$ and $u = x^2 - y^2, v = yx$

9. If $x = \sqrt{vw},\ y = \sqrt{wu},\ z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi, v = r\sin\theta\sin\phi, \ w = r\cos\theta$, calculate $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

10. If $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $v = \sin^{-1}x + \sin^{-1}y$, show that u & v are functionally dependent and find the functional relationship.

11. If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$, show that u & v are functionally dependent and find the functional relationship.

III. TAYLOR'S SERIES FOR FUNCTION OF TWO VARIABLES

$f(x,y) = f(a,b) + \Delta f(a,b) + \frac{\Delta^2 f(a,b)}{2!} + \frac{\Delta^3 f(a,b)}{3!} + \cdots\cdots$ where

$\Delta = (x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}$.

Examples

Expand the following functions upto second degree terms.

1. $f(x,y) = x^2 + xy + y^2$ in powers of $(x-1)$ and $(y-2)$.
2. $f(x,y) = (1+x+y)^{-1}$ in powers of $(x-1)$ and $(y-1)$.
3. $f(x,y) = e^x\cos y$ in powers of $(x-1)$ and $\left(y - \pi/4\right)$.
4. $f(x,y) = x^y$ in powers of $(x-1)$ and $(y-1)$ and also find $(1.1)^{1.1}$.
5. $f(x,y) = \tan^{-1}\left(y/x\right)$ in powers of $(x-1)$ and $(y-1)$. Hence compute $f(1.1, 0.9)$.

6. $f(x,y)=x^3+xy^2+y^3$ in powers of $(x-1)$ and $(y-2)$.

7. $f(x,y)=\cot^{-1}(xy)$ in powers of $(x+0.5)$ and $(y-2)$ and hence compute $f(-0.4,2.2)$.

8. $f(x,y)=x^2y+3y-2$ in powers of $(x+1)$ and $(y-2)$.

9. $f(x,y)=\sin(xy)$ in powers of $(x-1)$ and $\left(y-\pi/2\right)$

10. $f(x,y)=xy^2+\cos xy$ in powers of $(x-1)$ and $\left(y-\pi/2\right)$.

11. $f(x,y)=x^2y+\sin y+e^x$ about $(1,\pi)$.

12. $f(x,y)=e^x\sin y$ about $(-1,\pi/4)$.

V. MACLAURIN'S SERIES FOR FUNCTION OF TWO VARIABLES

$$f(x,y)=f(0,0)+\Delta f(0,0)+\frac{\Delta^2 f(0,0)}{2!}+\frac{\Delta^3 f(0,0)}{3!}+\cdots\cdots \text{ where } \Delta=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}.$$

Examples

1. $\cos x\cos y$ in powers of x and y up to second degree terms.

2. Expand $e^y\log_e(1+x)$ in powers of x and y up to third degree terms.

3. Expand $e^{ax}\sin by$ in the ascending powers of x and y up to third degree terms.

4. Find the Maclaurin's expansion of $e^x\log_e(1+y)$ up to third degree terms.

Expand $\log_e(1+x+y)$ in the neighbourhood of (0,0) up to second degree terms.

IV. MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

A function $z=f(x,y)$ is said to have a maximum or minimum at $x=a, y=b$ as $f(a+h,b+k)<f(a,b)$ or $f(a+h,b+k)>f(a,b)$ for all values of h and k.

Necessary conditions for $f(x,y)$ to have a maximum or a minimum at (a,b) are $f_x(a,b)=0$ and $f_y(a,b)=0$.

Working rule to find the maxima and minima of $z=f(x,y)$

1) Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ and equate them to zero, solve these as simultaneous equations in x and y. Let $(a,b),(c,d)$........... be the roots of the simultaneous equations.

2) Calculate the values of $r=\frac{\partial^2 z}{\partial x^2}, s=\frac{\partial^2 f}{\partial x\partial y}, t=\frac{\partial^2 f}{\partial y^2}$ for each of points.

3) If $rt-s^2>0$ and $r<0$ at (a,b), $f(x,y)$ has a maximum value.

4) If $rt - s^2 > 0$ and $r > 0$ at (a,b), $f(x,y)$ has a minimum value.

5) If $rt - s^2 < 0$ at (a,b), (a,b) is a saddle point.

6) If $rt - s^2 = 0$ at (a,b) then the case is doubtful and needs further investigation.

Examples

1. Find the extreme values of the following functions:
   a) $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.
   b) $f(x,y) = 2\left(x^2 - y^2\right) - x^4 + y^4$.
   c) $f(x,y) = xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$
   d) $x^4 + y^4 - x^2 - y^2 + 1$
   e) $x^2 + 2y^2 + 3z^2 - 2xy - 2yz - 2$
   f) $f(x,y) = x^3y^2(1 - x - y)$
   g) $f(x,y) = x^3 + y^3 - 3y - 12x + 20$
   h) $f(x,y) = \cos x \cos y \cos(x+y)$
   i) $x^2 + y^2 + 6x + 12 = 0$
   j) $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

2. Find the shortest distance from origin to the plane $x - 2y - 2z = 3$.

3. Find the points on the surface $z^2 = xy + 1$ nearest to the origin.

4. Find the shortest distance from origin to the surface $xyz^2 = 2$.

5. Sum of three numbers is a constant. Prove that their product is maximum when they are equal.

6. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.

7. Examine the function $f(x,y) = \sin x + \sin y + \sin(x+y)$, $x, y \in (0,\pi)$ for extreme values.

8. In a plane triangle find the maximum value of $\cos A \cos B \cos C$ where A, B and C are the angles of the triangle.

9. Find the minimum value of $x^2 + y^2 + z^2$ given $ax + by + cz = p$

10. A flat circular plate is heated so that the temperature at any point (x,y) is $u(x,y) = x^2 + 2y^2 - x$. Find the coldest point on the plate.

11. A rectangular box open at the top is to have a volume 108 cubic meters. Find its dimensions if its total surface area is minimum.

12. The temperature T at any point (x, y, z) in space is $T(x, y, z) = kxyz^2$ where k is a constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

V. GRADIENT DESCENT METHOD

Approximate the minimum point of the following functions near the given point using Gradient descent method (Perform three iterations)

1. $f(x, y) = 3x^2 + y^2$ near $(1, 3)$
2. $f(x, y) = x^2 - xy + y^2$ near $\left(1, \frac{1}{2}\right)$
3. $f(x, y) = 4x^2 - 4xy + 2y^2$ near $(0, 0)$
4. $f(x, y) = \left(\frac{3}{4}x - \frac{3}{2}\right)^2 + (y-2)^2 + \frac{1}{4}xy$ near $(5, 4)$
5. $f(x, y) = 4x^2 - 8xy + 6y^2$ near $(1, 1)$

$f(x, y) = x - y + 2x^2 + 2xy + y^2$ near $(0, 0)$