

BMS COLLEGE OF ENGINEERING, BENGALURU-19 Autonomous Institute, Affiliated to VTU

DEPARTMENT OF MATHEMATICS

Dept. of Maths., BMSCE

Unit - 2: Multivariable Calculus

For the Course Code: 23MA1BSMCS

I. PARTIAL DERIVATIVES

Let $u = f\left(x,y\right)$ be a function of two variables x and y. If we keep y as constant and vary x alone, then u is a function of x only. The derivative of u with respect to x, treating y as constant is called the partial derivative of u w. r. t x and is denoted by one of the symbols $\frac{\partial u}{\partial x}$, u_x , Thus $\frac{\partial u}{\partial x} = u_x = \lim_{\Delta x \to 0} \frac{f\left(x + \Delta x, y\right) - f\left(x, y\right)}{\Delta x}$. Similarly $\frac{\partial u}{\partial y} = u_y = \lim_{\Delta y \to 0} \frac{f\left(x, y + \Delta y\right) - f\left(x, y\right)}{\Delta y}$. Further u_x and u_y are also functions of x and y, so these can further be differentiated partially w. r. t. x and y. Thus $\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$ or u_{xx} , $\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$ or u_{xy} , $\frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$ or u_{yx} and $\frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$ or u_{yy} .

1. Prove that
$$yv_y - xv_x = -y^2v^3$$
 if $v = (1-2xy + y^2)^{-1/2}$.

2. Show that
$$w_x + w_y + w_z = 0$$
 if $w = (y - z)(z - x)(x - y)$.

3. Show that
$$u_x + u_y = u$$
, if $u = \frac{e^{x+y}}{e^x + e^y}$.

4. If
$$w = x^2y + y^2z + z^2x$$
, prove that $w_x + w_y + w_z = (x + y + z)^2$.

5. If
$$u = \frac{y}{z} + \frac{z}{x}$$
, show that $xu_x + yu_y + zu_z = 0$.

6. Given
$$u = e^{r\cos\theta}\cos(r\sin\theta)$$
, $v = e^{r\cos\theta}\sin(r\sin\theta)$, Prove that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}$$
 and $\frac{\partial v}{\partial r} = \frac{-1}{r} \frac{\partial u}{\partial \theta}$

7. If
$$(x+y)z = x^2 + y^2$$
, Show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$.

8. Show that
$$(v_x)^2 + (v_y)^2 + (v_z)^2 = v^4$$
 if $v = (x^2 + y^2 + z^2)^{-1/2}$.



Dept. of Maths., BMSCE

9. If
$$z = e^{ax + by} f(ax - by)$$
 prove that $b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2abz$.

10. If
$$u = x^2 \tan^{-1} \left(\frac{y}{x} \right) - y^2 \tan^{-1} \left(\frac{x}{y} \right)$$
, then show that

(i)
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$$
 and (ii) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

11. If
$$u = \log_e (x^3 + y^3 - x^2y - xy^2)$$
, then show that $u_{xx} + 2u_{xy} + u_{yy} = -\frac{4}{(x+y)^2}$.

12. If
$$z = f(x+ct) + \phi(x-ct)$$
, then prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$.

13. If
$$u = e^{a\theta} \cos(a \log_e r)$$
 Show that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$.

14. Prove that
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$
, if

a)
$$u = x^y$$

b)
$$u = \log_e \left(\frac{x^2 + y^2}{xy} \right)$$

15. Verify that
$$f_{xy} = f_{yx}$$
 when $f(x, y) = \sin^{-1}(y/x)$.

16. Prove that
$$u_{xx} + u_{yy} = 0$$
, if

a)
$$u = \tan^{-1} \left(\frac{2xy}{x^2 - y^2} \right)$$

b)
$$u = \log_e (x^2 + y^2) + \tan^{-1} (y/x)$$

17. Find the value of *n* so that the equation $v = r^n (3\cos^2 \theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial v}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v}{\partial \theta} \right) = 0.$$

18. If
$$Z = \tan(y + ax) + (y - ax)^{3/2}$$
, Show that $Z_{xx} = a^2 Z_{yy}$.

19. If
$$v = \frac{1}{\sqrt{t}} e^{-x^2/4a^2t}$$
, Prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$.

20. If
$$\theta = t^n e^{-r^2/4t}$$
, what value of *n* will make $\frac{1}{r^2} \left[\frac{\partial}{\partial r} \left(r^2 \frac{\partial \theta}{\partial r} \right) \right] = \frac{\partial \theta}{\partial t}$.

21. If
$$x^x y^y z^z = c$$
, show that $z_{xy} = -[x \log_e(ex)]^{-1}$, when $x = y = z$.

22. If
$$z = \log_e(e^x + e^y)$$
, Show that $rt - s^2 = 0$ where $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$.

23. If
$$v = \log_e(x^2 + y^2 + z^2)$$
, prove that $(x^2 + y^2 + z^2) \left[v_{xx} + v_{yy} + v_{zz} \right] = 2$.

24. If
$$u = \log_e \left[x^3 + y^3 + z^3 - 3xyz \right]$$
, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = \frac{-9}{\left(x + y + z \right)^2}$.



Dept. of Maths., BMSCE

25. If
$$w = r^m$$
, prove that $w_{xx} + w_{yy} + w_{zz} = m(m+1)r^{m-2}$ where $r^2 = x^2 + y^2 + z^2$.

- 26. Verify that v satisfies Laplace's equation $v_{xx} + v_{yy} + v_{zz} = 0$ if
 - a) $v = e^{3x+4y}\cos 5z$
 - b) $v = (x^2 + y^2 + z^2)^{-1/2}$
 - c) $v = \cos 3x \cos 4y \sinh 5z$

II. <u>COMPOSITE FUNCTIONS</u>

Total Derivative:

If
$$u = f(x, y)$$
 where $x = \phi(t)$ and $y = \psi(t)$ then $\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$.

Examples

- 1. Find the differential of the following functions:
 - a) $f(x, y) = x \cos y y \cos x$
 - 2. $f(x, y) = e^{xyz}$
 - 3. Find $\frac{du}{dt}$ for the following functions:

a)
$$u = x^2 - y^2$$
, $x = e^t \cos t$, $y = e^t \sin t$ at $t = 0$.

b)
$$u = x^y$$
, when $y = \tan^{-1} t$, $x = \sin t$

(c)
$$u = \sin(e^x + y)$$
, $x = f(t)$, $y = g(t)$

d)
$$u = \tan^{-1}(y/x), x = e^t - e^{-t}, y = e^t + e^{-t}$$

$$u = xy + yz + zx, x = 1/t, y = e^{t} \text{ and } z = e^{-t}$$

f)
$$u = x^3 y e^z$$
 where $x = t$, $y = t^2$ and $z = \log_e t$, at $t = 2$.

4. Find $\frac{du}{dt}$ and verify the result by direct substitution.

a)
$$u = \sin\left(\frac{x}{y}\right)$$
 and $x = e^t$, $y = t^2$.

b)
$$u = x^2 + y^2 + z^2$$
, $x = e^{2t}$, $y = e^{2t} \cos 3t$ and $z = e^{2t} \sin 3t$

- 5. If $u = \sin^{-1}(x y)$, x = 3t, $y = 4t^3$, show that $\frac{du}{dt} = 3(1 t^2)^{-\frac{1}{2}}$. Also verify the result by direct substitution.
- 6. The altitude of a right circular cone is 15cm and is increasing at 0.2 cm/s. The radius of the base is 10cm and is decreasing at 0.3 cm/s. How fast is the volume changing?



- 7. Find the rate at which the area of a rectangle is increasing at a given instant when the sides of the rectangle are 4 ft and 3 ft and are increasing at the rate of 1.5 fts⁻¹ and 0.5 fts⁻¹ respectively.
- 8. In order that the function $u = 2xy 3x^2y$ remains constant, what should be the rate of change of y w.r.t. t, given x increases at the rate of 2cm/sec at the instant when x = 3cm and y = 1cm.

Partial differentiation of Composite functions:

If
$$u = f(x, y)$$
 where $x = g(s, t)$ and $y = h(s, t)$ then
$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} \text{ and } \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$$

Examples

1. If
$$u = x^2 - y^2$$
, $x = 2r - 3s + 4$, $y = -r + 8s - 5$ find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$.

2. If
$$W = u^2 v$$
 and $u = e^{x^2 - y^2}$, $v = \sin(xy^2)$ find $\frac{\partial W}{\partial x}$ and $\frac{\partial W}{\partial y}$.

3. If
$$u = u \left[\frac{y - x}{xy}, \frac{z - x}{xz} \right]$$
, show that $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$.

4. If
$$u = F(x - y, y - z, z - x)$$
, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

5. Prove that
$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$$
 if z is a function of x and y and $x = e^{u} + e^{-v}$, $y = e^{-u} - e^{v}$.

6. If
$$V = f(r, s, t)$$
 and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$ show that $x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} = 0$.

7. If
$$x = u + v + w$$
, $y = vw + wu + uv$, $z = uvw$ and F is a function of x , y , z show that
$$u \frac{\partial F}{\partial u} + v \frac{\partial F}{\partial v} + w \frac{\partial F}{\partial w} = x \frac{\partial F}{\partial x} + 2y \frac{\partial F}{\partial y} + 3z \frac{\partial F}{\partial z}$$

8. If z is a function of x and y and $x = e^u \cos v$, $y = e^u \sin v$.

Prove that (i)
$$x \frac{\partial z}{\partial v} + y \frac{\partial z}{\partial u} = e^{2u} \frac{\partial z}{\partial y}$$
 (ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u} \left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$



If u and v are functions of two independent variable x and y then the Jacobian of u, v w.r.to x, y is denoted by

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ or } J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Properties of Jacobians

1. If u, v are functions of x, y and x, y are functions of u, v

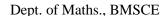
$$J = \frac{\partial(u,v)}{\partial(x,y)}$$
 and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$.

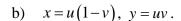
2. If u, v are functions of r, s and r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}.$$

Examples

- 1. Find $\frac{\partial(u,v)}{\partial(x,y)}$ for the following:
 - a) $u = e^x \sin y, \ v = x + \log_e (\sin y).$
 - b) u = x + y, y = uv
- 2. If $x = a \cosh \xi \cos \eta$, $y = a \sinh \xi \sin \eta$ Show that $\frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{1}{2}a^2(\cosh 2\xi \cos 2\eta)$.
- 3. Find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ for the following:
 - a) $u = x^2$, $v = \sin y$, $w = e^{-3z}$.
 - b) $u = x^2 + y^2 + z^2$, v = xy + yz + zx, w = x + y + z.
 - c) $u = x + 3y^2 z^3, v = 4x^2yz, w = 2z^2 xy$ at (1,-1,0).
 - d) $u = \frac{2yz}{x}, v = \frac{3zx}{y}, w = \frac{4xy}{z}$
- 4. If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$, then show that $\frac{\partial (y_1, y_2, y_3)}{\partial (x_1, x_2, x_3)} = 4$.
- 5. Find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$ if
 - a) $x = \frac{u^2 v^2}{2}$, y = uv, z = w.
 - b) u = x + y + z, uv = y + z, uvw = z,
- 6. Verify that JJ'=1
 - a) $x = e^u \cos v$, $y = e^u \sin v$.





c)
$$u = x + \frac{y^2}{x}, v = \frac{y^2}{x}$$
.

d)
$$x = u, y = u \tan v, z = w$$

7. Find $\frac{\partial(u,v)}{\partial(r,\theta)}$ when,

a)
$$u = x^2 - 2y^2$$
, $v = 2x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$

b)
$$u = x^2 - y^2$$
, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$

c)
$$u = 2xy, v = x^2 - y^2$$
 and $x = r\cos\theta, y = r\sin\theta$

d)
$$u = 2axy \& v = a(x^2 - y^2)$$
 and $x = r\cos\theta$, $y = r\sin\theta$

8. Find
$$\frac{\partial(X,Y)}{\partial(x,y)}$$
 where $X = u^2v$, $Y = uv^2$ and $u = x^2 - y^2$, $v = yx$

9. If
$$x = \sqrt{vw}$$
, $y = \sqrt{wu}$, $z = \sqrt{uv}$ and $u = r \sin \theta \cos \phi$, $v = r \sin \theta \sin \phi$, $w = r \cos \theta$, calculate $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$.

10. If
$$u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$
, $v = \sin^{-1} x + \sin^{-1} y$, show that $u \& v$ are functionally dependent and find the functional relationship.

11. If
$$u = \frac{x+y}{1-xy}$$
, $v = \tan^{-1} x + \tan^{-1} y$, show that $u \& v$ are functionally dependent and find the functional relationship.

III. TAYLOR'S SERIES FOR FUNCTION OF TWO VARIABLES

$$f(x,y) = f(a,b) + \Delta f(a,b) + \frac{\Delta^2 f(a,b)}{2!} + \frac{\Delta^3 f(a,b)}{3!} + \dots \text{ where}$$

$$\Delta = (x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}.$$

Examples

Expand the following functions upto second degree terms.

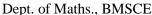
1.
$$f(x, y) = x^2 + xy + y^2$$
 in powers of $(x-1)$ and $(y-2)$.

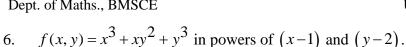
2.
$$f(x, y) = (1 + x + y)^{-1}$$
 in powers of $(x-1)$ and $(y-1)$

3.
$$f(x,y) = e^x \cos y$$
 in powers of $(x-1)$ and $\left(y - \frac{\pi}{4}\right)$.

4.
$$f(x, y) = x^y$$
 in powers of $(x-1)$ and $(y-1)$ and also find $(1.1)^{1.1}$.

5.
$$f(x, y) = \tan^{-1} \left(\frac{y}{x} \right)$$
 in powers of $(x-1)$ and $(y-1)$. Hence compute $f(1.1, 0.9)$.





- 7. $f(x, y) = \cot^{-1}(xy)$ in powers of (x+0.5) and (y-2) and hence compute f(-0.4, 2.2).
- 8. $f(x, y) = x^2y + 3y 2$ in powers of (x+1) and (y-2).
- 9. $f(x,y) = \sin(xy)$ in powers of (x-1) and $(y-\pi/2)$
- 10. $f(x, y) = xy^2 + \cos xy$ in powers of (x-1) and $(y-\pi/2)$.
- 11. $f(x, y) = x^2 y + \sin y + e^x$ about $(1, \pi)$.
- 12. $f(x, y) = e^x \sin y$ about $(-1, \pi/4)$.

V. MACLAURIN'S SERIES FOR FUNCTION OF TWO VARIABLES

$$f(x,y) = f(0,0) + \Delta f(0,0) + \frac{\Delta^2 f(0,0)}{2!} + \frac{\Delta^3 f(0,0)}{3!} + \dots \quad \text{where } \Delta = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.$$

Examples

- $\cos x \cos y$ in powers of x and y up to second degree terms.
- **2.** Expand $e^y \log_e (1+x)$ in powers of x and y up to third degree terms.
- Expand $e^{ax} \sin by$ in the ascending powers of x and y up to third degree terms.
- Find the Maclaurin's expansion of $e^x \log_e (1+y)$ up to third degree terms.

Expand $\log_{e}(1+x+y)$ in the neighbourhood of (0,0) up to second degree terms.

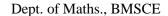
IV. MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

A function z = f(x, y) is said to have a maximum or minimum at x = a, y = b as f(a+h,b+k) < f(a,b) or f(a+h,b+k) > f(a,b) for all values of h and k.

Necessary conditions for f(x,y) to have a maximum or a minimum at (a,b) are $f_{\chi}(a,b) = 0$ and $f_{\chi}(a,b) = 0$.

Working rule to find the maxima and minima of z = f(x, y)

- 1) Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ and equate them to zero, solve these as simultaneous equations in x and y. Let (a,b),(c,d)..... be the roots of the simultaneous equations.
- 2) Calculate the values of $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ for each of points.
- 3) If $rt-s^2 > 0$ and r < 0 at (a,b), f(x,y) has a maximum value.





- 4) If $rt-s^2>0$ and r>0 at (a,b), f(x,y) has a minimum value.
- 5) If $rt-s^2 < 0$ at (a,b),(a,b) is a saddle point.
- 6) If $rt s^2 = 0$ at (a,b) then the case is doubtful and needs further investigation.

Examples

1. Find the extreme values of the following functions:

a)
$$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$
.

b)
$$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$$
.

c)
$$f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

d)
$$x^4 + y^4 - x^2 - y^2 + 1$$

e)
$$x^2 + 2y^2 + 3z^2 - 2xy - 2yz - 2$$

f)
$$f(x, y) = x^3y^2(1-x-y)$$

g)
$$f(x, y) = x^3 + y^3 - 3y - 12x + 20$$

h)
$$f(x, y) = \cos x \cos y \cos(x + y)$$

i)
$$x^2 + y^2 + 6x + 12 = 0$$

j)
$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

- 2. Find the shortest distance from origin to the plane x 2y 2z = 3.
- 3. Find the points on the surface $z^2 = xy + 1$ nearest to the origin.
- 4. Find the shortest distance from origin to the surface $xyz^2 = 2$.
- 5. Sum of three numbers is a constant. Prove that their product is maximum when they are equal.
- 6. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.
- 7. Examine the function $f(x, y) = \sin x + \sin y + \sin(x + y)$, $x, y \in (0, \pi)$ for extreme values.
- 8. In a plane triangle find the maximum value of $\cos A \cos B \cos C$ where A, B and C are the angles of the triangle.
- 9. Find the minimum value of $x^2 + y^2 + z^2$ given ax + by + cz = p
- 10. A flat circular plate is heated so that the temperature at any point (x, y) is $u(x, y) = x^2 + 2y^2 x$. Find the coldest point on the plate.
- 11. A rectangular box open at the top is to have a volume 108 cubic meters. Find its dimensions if its total surface area is minimum.



12. The temperature T at any point (x, y, z) in space is $T(x, y, z) = kxyz^2$ where k is a constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

V. GRADIENT DESCENT METHOD

Approximate the minimum point of the following functions near the given point using Gradient descent method (Perform three iterations)

1.
$$f(x,y) = 3x^2 + y^2$$
 near (1,3)

2.
$$f(x,y) = x^2 - xy + y^2 \text{ near } \left(1, \frac{1}{2}\right)$$

3.
$$f(x, y) = 4x^2 - 4xy + 2y^2$$
 near $(0, 0)$

4.
$$f(x,y) = \left(\frac{3}{4}x - \frac{3}{2}\right)^2 + \left(y - 2\right)^2 + \frac{1}{4}xy$$
 near (5,4)

5.
$$f(x, y) = 4x^2 - 8xy + 6y^2$$
 near (1,1)

$$f(x, y) = x - y + 2x^2 + 2xy + y^2$$
 near $(0,0)$