B.M.S. COLLEGE OF ENGINEERING,
BENGALURU,DEPARTMENT OF MATHEMATICS

For the Course Code: 22MA1BSMCV, 22MA1BSMES, 22MA1BSMME & 22MA1BSMCS

I. RANK OF MATRICES

1. Find the rank of the following matrices by reducing them into echelon form.

II.

a) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$ Ans: $\rho=2$.

b) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ Ans: $\rho=2$.

c) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ Ans: $\rho=3$.

d) $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$ Ans: $\rho=3$.

e) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$ Ans: $\rho=2$.

f) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ Ans: $\rho=2$.

g) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ Ans: $\rho=2$.

h) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ Ans: $\rho=3$.

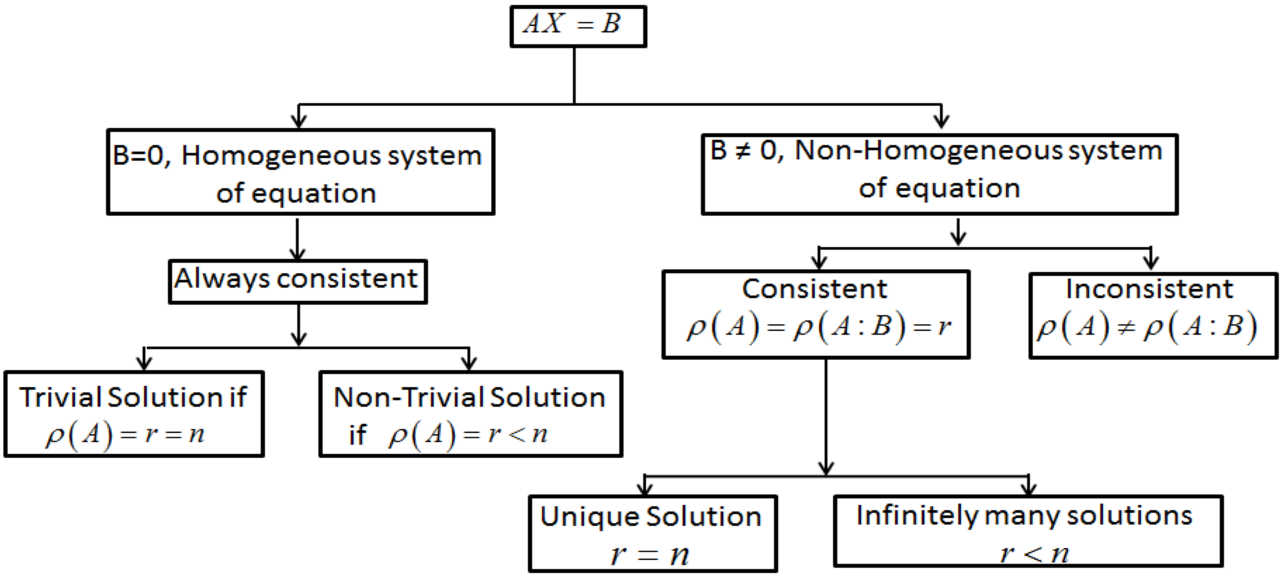
i) $\begin{bmatrix} 1 & -2 \\ 3 & -6 \\ 7 & -1 \\ 4 & 5 \end{bmatrix}$

j) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$ Ans: $\rho=2$.

k) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$ Ans: $\rho=4$.

1. Find 'b' if the rank of $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$ is 3. Ans: $b=2$ or $b=-6$.

III. Consistency and solution of linear system of equations:



$n =$ Number of Unknowns

1. Discuss the consistency of the following system of linear equations

a)
$$x+2y+3z=0$$
$$3x+4y+4z=0$$
$$7x+10y+12z=0$$

Ans: Consistent and has trivial solution.

b)
$$2x_1+3x_2-4x_3+x_4=0$$
$$x_1-x_2+x_3+2x_4=0$$
$$5x_1-x_3+7x_4=0$$
$$7x_1+8x_2-11x_3+5x_4=0$$

Ans: Consistent and has infinitely many solutions

$$x_1=\frac{(k_1-7k_2)}{5}, x_2=\frac{(6k_1+3k_2)}{5}, x_3=k_1, x_4=k_2$$

c)
$$5x+3y+7z=4$$
$$3x+26y+2z=9$$
$$7x+2y+10z=5$$

Ans: Consistent and has infinitely many solutions

$$x=\frac{7-16k}{11} \quad y=\frac{k+3}{11} \quad z=k$$

d)
$$4x-2y+6z=8$$
$$x+y-3z=-1$$
$$15x-3y+9z=21$$

Ans: Consistent and has infinitely many solutions

$$x=1,\ y=3k-2,\ z=k$$

e)
$$2x-3y+7z=5$$
$$3x+y-3z=13$$
$$2x+19y-47z=32$$

Ans: inconsistent

f)
$$2x_1+3x_2-x_3=1$$
$$3x_1-4x_2+3x_3=-1$$
$$2x_1-x_2+2x_3=-3$$
$$3x_1+1x_2-2x_3=4$$

Ans: Inconsistent

$x+2y+z=3$
$2x+3y+2z=5$
$3x-5y+5z=2$
$3x+9y-z=4$
Ans : Consistent and has unique solution

g) $x=-1,\ y=1,\ z=2$

$2x+6y+11=0$
$6x+20y-6z+3=0$
$6y-18z+1=0$

h) Ans: inconsistent

$2x_1-2x_2+4x_3+3x_4=9$
$x_1-x_2+2x_3+2x_4=6$
$2x_1-2x_2+x_3+2x_4=3$
$x_1-x_2+x_4=2$

i) Ans: Inconsistent

$2x+y-z=0$
$2x+5y+7z=52$
$x+y+z=9$
Ans : Consistent and has unique solution

j) $x=1, y=3, z=5$

$3x+2y+2z=0$
$x+2y=4$
$10y+3z=-2$
$2x-3y-z=5$
Ans : Consistent and has unique solution

k) $x=2, y=1, z=-4$

$x_1+x_2-2x_3+x_4+3x_5=1$
$2x_1-x_2+2x_3+2x_4+6x_5=2$
$3x_1+2x_2-4x_3-3x_4-9x_5=3$
Ans : Consistent and
has infinitely many solutions

l) $x_1=1,\ x_2=2a,\ x_3=a,\ x_4=-3b,\ x_5=b$

m)

2. Investigate the values of λ and μ so that the equations

$$2x+3y+5z=9$$
$$7x+3y-2z=8$$
$$2x+3y+\lambda z=\mu$$

have (i) no solution (ii) unique solution (iii) infinite number of solutions.

Ans: (i) If $\lambda=5$ and $\mu\neq 9$ (ii) $\lambda\neq 5$ and μ can be any value (iii) $\lambda=5$ and $\mu=9$.

3. Determine the values of a and b for which the system

$$x+2y+3z=6$$
$$x+3y+5z=9$$
$$2x+5y+az=b$$

have (i) no solution (ii) unique solution (iii) infinite number of solutions.

Ans: (i) If $a=8$ and $b\neq 15$ (ii) $a\neq 8$ and for any b (iii) $a=8$ and $b=15$.

4. Find the value of a for which the system $x+2y+z=3$; $ay+5z=10$; $2x+7y+az=b$ has a unique solution. Also find the pair of values (a,b) for which it has infinitely many solutions.

Ans:

5. Find the values of a and b for which the system of equations

$$x+ay+z=3$$
$$x+2y+2z=b$$
$$x+5y+3z=9$$

is consistent.

Ans: If $a=-1$ and b=6 equations will be consistent and have infinite number of solutions.

If $a\neq -1$ and b has any value, equations will be consistent and have a unique solution.

6. Show that the equations

$$3x+4y+5z=a$$
$$4x+5y+6z=b$$
$$5x+6y+7z=c$$

do not have a solution unless $a+c=2b$.

7. Test for consistency

$$-2x+y+z=a$$
$$x-2y+z=b$$
$$x+y-2z=c$$

where a,b and c are constants.

Ans: if $a+b+c\neq 0$ inconsistent, if $a+b+c=0$, then infinitely many solution.

IV. Gauss elimination method:

1. Solve the following system of equations by Gauss elimination method:

a)
$$2x_1+x_2+x_3=10$$
$$3x_1+2x_2+3x_3=18$$
$$x_1+4x_2+9x_3=16$$
Ans: $x_1=7,\ x_2=-9,\ x_3=5.$

b)
$$2x+2y+z=12$$
$$3x+2y+2z=8$$
$$5x+10y-8z=10$$
Ans: $x=\frac{-51}{4}, y=\frac{115}{8}, z=\frac{35}{4}$

c)
$$2x_1+4x_2+x_3=3$$
$$3x_1+2x_2-2x_3=-2$$
$$x_1-x_2+x_3=6$$
Ans: $x_1=2,\ x_2=-1,\ x_3=3.$

d)
$$10x+2y+z=9$$
$$2x+20y-2z=-44$$
$$-2x+3y+10z=22$$
Ans: $x=1, y=-2, z=3$

e)
$$2x_1+x_2+4x_3=12$$
$$8x_1-3x_2+2x_3=20$$
$$4x_1+11x_2-x_3=33$$
Ans: $x_1=3,\ x_2=2,\ x_3=1.$

f)
$$x_1+4x_2-x_3=-5$$
$$x_1+x_2-6x_3=-12$$
$$3x_1-x_2-x_3=4$$
Ans: $x_1=\frac{117}{71},\ x_2=-\frac{81}{71},\ x_3=\frac{148}{71}.$

g)
$$2x_1+x_2+3x_3=1$$
$$4x_1+4x_2+7x_3=1$$
$$2x_1+5x_2+9x_3=3$$
Ans: $x_1=-1/2,\ x_2=-1,\ x_3=1.$

h)
$$2x_1-7x_2+4x_3=9$$
$$x_1+9x_2-6x_3=1$$
$$-3x_1+8x_2+5x_3=6$$
Ans: $x_1=4, x_2=1, x_3=2$

$$\begin{aligned} 5x_1 + x_2 + x_3 + x_4 &= 4 \\ x_1 + 7x_2 + x_3 + x_4 &= 12 \\ x_1 + x_2 + 6x_3 + x_4 &= -5 \\ x_1 + x_2 + x_3 + 4x_4 &= -6 \end{aligned}$$

i) Ans: $x_1 = 1,\ x_2 = 2,\ x_3 = -1,\ x_4 = -2$

2. Show that if $\lambda \neq -5$ the system of equations $\begin{aligned} 3x - y + 4z &= 3 \\ x + 2y - 3z &= -2 \\ 6x + 5y + \lambda z &= -3 \end{aligned}$ have a unique solution. If $\lambda = -5$ show that the equations are consistent. Determine the solution in each case.

Ans: when $\lambda \neq -5,\ x = \frac{4}{7},\ y = \frac{-9}{7},\ z = 0$, when $\lambda = -5,\ x = \frac{4-5k}{7},\ y = \frac{13k-9}{7},\ z = k$.

3. Prove that the equations $\begin{aligned} 5x + 3y + 2z &= 12 \\ 2x + 4y + 5z &= 2 \\ 39x + 43y + 45z &= c \end{aligned}$ are incompatible unless $c = 74$; and in that case the equations are satisfied by $x = 2 + t, y = 2 - 3t, z = -2 + 2t$, where t is any arbitrary quantity.

4. For what values of k the equations $\begin{aligned} x + y + z &= 1 \\ 2x + y + 4z &= k \\ 4x + y + 10z &= k^2 \end{aligned}$ have a solution and solve them completely in each case. **Ans**: when $k = 1,\ x = -3z, y = 2z + 1$, when $k = 2$, $x = 1 - 3z, y = 2z.$

V. Iterative methods:

Gauss-Seidel Iteration method

The solution converges if the system is diagonally dominant.

Suppose $AX = B$ is diagonally dominant with $A = (a_{ij})$, $X = (x_i)$ and $B = (b_i)$.

$$x_1^{n+1} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^n - a_{13}x_3^n \cdots\cdots\right)$$
$$x_2^{n+1} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{n+1} - a_{23}x_3^n \cdots\cdots\right)$$
$$x_3^{n+1} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{n+1} - a_{32}x_2^{n+1} \cdots\cdots\right)$$
$$\vdots$$

Then the iterative formula is

a.
$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$
Ans: $x = 1,\ y = -1,\ z = 1$.

b.
$$5x + 2y + z = 12$$
$$x + 4y + 2z = 15$$
$$x + 2y + 5z = 20$$
Ans: $x = 0.996,\ y = 2,\ z = 3$.

c.
$$2x + y + 6z = 9$$
$$8x + 3y + 2z = 13$$
$$x + 5y + z = 7$$
Ans: $x = 1,\ y = 1,\ z = 1$.

d.
$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$
Ans: $x = 0.9876,\ y = 1.5090,\ z = 1.8485$

.

e.
$$10x + 2y + z = 9$$
$$2x + 20y - 2z = -44$$
$$-2x + 3y + 10z = 22$$
Ans: $x = 1,\ y = -2,\ z = 3$.

f.
$$83x + 11y - 4z = 95$$
$$7x + 52y + 13z = 104$$
$$3x + 8y + 29z = 71$$
Ans: $x = 1.06, y = 1.37, z = 1.96$.

g.
$$54x + y + z = 110$$
$$2x + 15y + 6z = 72$$
$$-x + 6y + 27z = 85$$
Ans: $x = 1.926,\ y = 3.573,\ z = 2.425$.

h.
$$5x_1 - x_2 = 9$$
$$-x_1 + 5x_2 - x_3 = 4$$
$$-x_2 + 5x_3 = -6$$
Ans: $x_1 = 1.99,\ x_2 = 0.99,\ x_3 = -1$.

i.
$$8x_1 + x_2 - x_3 = 8$$
$$2x_1 + x_2 + 9x_3 = 12$$
$$x_1 - 7x_2 + 2x_3 = -4$$
Ans: $x_1 = 1\ x_2 = 1,\ x_3 = 1$.

j.
$$4x_1 + 2x_2 + x_3 = 11$$
$$-x_1 + 2x_2 = 3$$
$$2x_1 + x_2 + 4x_3 = 16$$
Ans: $x_1 = 1,\ x_2 = 2, x_3 = 3$.

k. Start with $(2,2,-1)$ and solve
$$5x_1 - x_2 + x_3 = 10$$
$$2x_1 + 4x_2 = 12$$
$$x_1 + x_2 + 5x_3 = -1$$
$Ans: x_1 = 2.5555, x_2 = 1.7222,\ x_3 = -1.0555$

l.
$$10x_1 + x_2 + x_3 = 12$$
$$2x_1 + 10x_2 + x_3 = 13$$
$$2x_1 + 2x_2 + 10x_3 = 14$$
Ans: $x_1 = x_2 = x_3 = 1$.

m.
$$27x_1 + 6x_2 - x_3 = 85$$
$$6x_1 + 15x_2 + 2x_3 = 72$$
$$x_1 + x_2 + 54x_3 = 110$$
Ans: $x_1 = 2.4255,\ x_2 = 3.573,\ x_3 = 1.926$.

n.
$$x_1 - 8x_2 + 3x_3 = -4$$
$$2x_1 + x_2 + 9x_3 = 12$$
$$8x_1 + 2x_2 - 2x_3 = 8$$
$Ans: x_1 = x_2 = x_3 = 1$.

o.
$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$
Ans: $x = 1,\ y = 2,\ z = 3, u = 4$.

VI. Characteristic values (Eigen values) and characteristic vectors (Eigen vectors)

If A is a square matrix, then λ is said to be an eigen value of the matrix if there exists a non-zero vector X such that $AX = \lambda X$. X is called the eigen vector corresponding to the eigen value λ.

$\lambda X = \lambda IX \Rightarrow (A - \lambda I)X = 0$. We seek non-trivial solution of $(A - \lambda I)X = 0$.

X is non-trivial if $\rho(A - \lambda I) < n \Rightarrow |A - \lambda I| = 0$.

If A is matrix of size 3×3 then $-\lambda^3 + Tr(A)\lambda^2 - \sum M_{ii}^{2\times 2}\lambda + |A| = 0$.

a. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ Ans: $\lambda = -2, 3, 6; x_1 = [-k, 0, k],$ $x_2 = [k, -k, k],\ x_3 = [k, 2k, k]$.

b. $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ Ans: $\lambda = 2, 3, 5\ \ x_1 = k_1[1, -1, 0]$ $x_2 = k_2[1, 0, 0],\ x_3 = k_3[3, 2, 1]$

c. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ Ans: $\lambda = 0, 3, 15\ \ x_1 = [1, 2, 2],$ $x_2 = [2, 1, -2], x_3 = [2, -2, 1]$

d. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ Ans: $\lambda = 1, 2, 3\ \ x_1 = [1, 0, -1]$ $x_2 = [0, 1, 0], x_3 = [1, 0, 1]$

e. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ Ans: $\lambda = 5, -3, -3\ \ x_1 = k[1, 2, -1]$ $x_2 = [3k_1 - 2k_2, k_2, k_1]$.

f. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ Ans: $\lambda = 8, 2, 2; x_1 = [2, -1, 1],$ $x_2 = [1, 0, -2], x_3 = [1, 2, 0]$.

g. $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ Ans: $\lambda = 1, -1, 4;\ x_1 = [2, -1, 1],$ $x_2 = [0, 1, 1], x_3 = [-1, -1, 1]$.

h. $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ Ans: $\lambda = 1, 2, 2;\ x_1 = k_1[1, 1, -1],$ $x_2 = k_2[2, 1, 0], x_3 = k_3[2, 1, 0]$.

i. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ Ans: $\lambda = 5, 1, 1;\ x_1 = [1, 1, 1],$ $x_2 = [1, 0, -1], x_3 = [2, -1, 0]$.

j. $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ Ans: $\lambda = 2, 2, 3$ $x_1 = x_2 = [5, 2, -3], x_3 = [1, 1, -2]$

k. $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ Ans: $\lambda = 2, 2, -2$ $x_1 = x_2 = [0, 1, 1]\ \ x_3 = [-4, -1, 1]$

l. $\begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix}$ Ans: $\lambda = -5, 2, 2$ $x_1 = [3, 2, 4],\ x_2 = x_3 = [1, 3, -1]$

Dominant Eigenvalues and eigenvectors

(Rayleigh-Power method)

.

APPLICATIONS

1. Find the traffic flow in the net of one-way streets directions shown in the figure