B. M. S. COLLEGE OF ENGINEERING, BENGALURU – 560019
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DEPARTMENT OF MATHEMATICS
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Course: *Mathematical Foundation for Computer Science Stream-II (23MA2BSMCS)*
Mathematical Foundation for Mechanical and Civil Engineering Stream -II (23MA2BSMCM)
Mathematical Foundation for Electrical Stream-II (23MA2BSMES)

UNIT-4: NUMERICAL METHODS-1

Solution of Algebraic and transcendental equations:

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n,\ \ a_0 \neq 0$ is called a polynomial of degree 'n' and the polynomial $f(x) = 0$ is called an algebraic equation of n^{th} degree. If $f(x)$ contains trigonometric, logarithmic or exponential functions, then $f(x) = 0$ is called a transcendental equation. For example, $x^2 + 2\sin x + e^x = 0$ is a transcendental equation.

- **Intermediate Value Theorem:**

If f is a function which is continuous at every point of the interval $[a,b]$ with $f(a)$ and $f(b)$ having opposite signs, then there exists at least one root between 'a' and 'b'.

- **Newton-Raphson method**

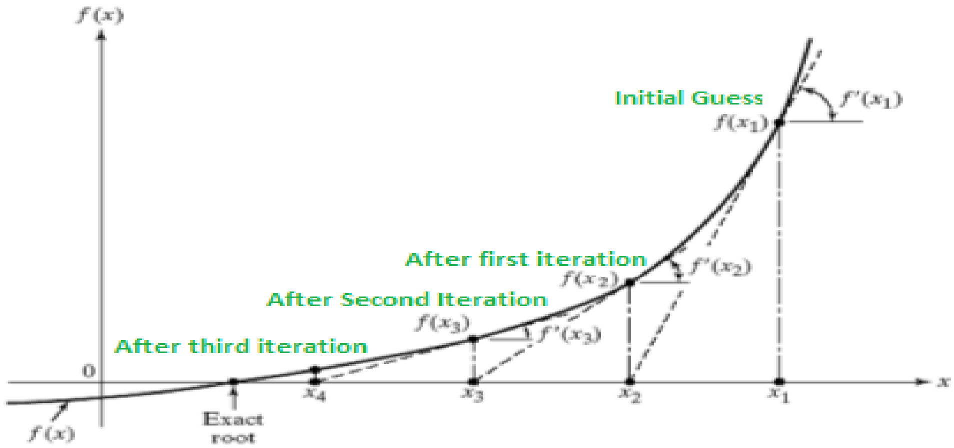

Figure: Typical iteration of NR method. (Differs from problem to problem)

The iterative formula is $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

If a root exists in the interval (a,b) then $x_0 = a$ or $x_0 = b$

or $x_0 =$ any point between a and b

or $x_0 = \dfrac{a+b}{2}$ could be taken as the initial approximation.

For $n = 1,\ x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ → 1st approximation/iteration

For $n = 2,\ x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$ → 2nd approximation/iteration

⋮

The iterations can be continued like this until desired accuracy.

1. Approximate the roots of the following equations by Newton-Raphson method

   a. $x^3 - 3x + 1 = 0$ near a) $x = 0.5$. **Ans**: 0.347 b) $x = 2$ **Ans**: 1.532
   b. $x\log_{10} x = 1.2$ in $(2,3)$ correct to 4 decimal places. **Ans**: 2.7406
   c. $3x = \cos x + 1$ in the interval $(0,1)$ correct to 4 decimal places. **Ans**: 0.6071
   d. $xe^x - 2 = 0$ near $x = 0.5$ **Ans**: 0.853
   e. $3\sin x - 2x + 5 = 0$ near $x = 3$. **Ans**: 0.684
   f. $x\sin x + \cos x = 0$ near $x = \pi$. **Ans**: 2.7985

2. Find the fourth root of 32 using Newton-Raphson method. (Hint: $x^4 - 32 = 0$). **Ans**: 2.3784

Finite differences and interpolation for equal intervals

Forward difference: $\Delta y_n = y_{n+1} - y_n$

Backward difference: $\nabla y_n = y_n - y_{n-1}$

k^{th} forward difference: $\Delta^k y_n = \Delta^{k-1} y_{n+1} - \Delta^{k-1} y_n$

k^{th} backward difference: $\nabla^k y_n = \nabla^{k-1} y_n - \nabla^{k-1} y_{n-1}$

Interpolation: Newton's forward and backward interpolation:

- **Newton's Forward Interpolation Formula**

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \ldots\ldots$$

where $x = x_0 + ph$.

- **Newton's Backward Interpolation Formula**

$$y = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \frac{q(q+1)(q+2)(q+3)}{4!}\nabla^4 y_n + \ldots\ldots$$

where $x = x_n + qh$.

Problems on Newton's Forward and Backward Interpolation:

1. The following data gives the melting point of an alloy of lead and zinc, where t is the temperature in C and p is the percentage of lead in the alloy.

$p(\%)$	60	70	80	90
t	226	250	276	304

Find the melting point of the alloy containing 84% of lead, using Newton's interpolation formula.

Ans: t=286.96 at p=84%

2. The current in a wire is measured with great precision as a function of time:

t	0	0.1250	0.25	0.375	0.5
i	0	6.24	7.75	4.85	0

Determine i at $t = 0.23s$.

Ans: i=7.8386 at t=0.23s

3. Using Newton's forward formula, compute the pressure of the steam at temperature 142^0 from the following steam table.

Temperature	140	150	160	170	180
Pressure	3.685	4.854	6.302	8.076	10.225

Ans: 3.8987

4. The following table gives the values of $\tan x$ for $0.10 \le x \le 0.30$. Find $\tan(0.26)$.

x	0.10	0.15	0.20	0.25	0.30
$\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Ans: tan(0.26) is 0.2659

5. The area of a circle (A) corresponding to diameter (D) is given below:

D	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find the area corresponding to diameter 105 using an appropriate interpolation formula.

Ans: 8666

6. Distance in nautical miles of the visible horizon for given heights in metres above the surface of the earth are given by the following table:

$x(heights)$	100	150	200	250	300	350	400
$y(distance)$	12	16	21	27	36	50	72

Find the value of y when $x = 225$ metres. Ans: 23.7588

7. Extrapolate for 25.4 given the data

x	19	20	21	22	23
y	91	100.25	110	120.25	131

Ans: 158.84

8. From the following table find the number of students who obtained less than 45 marks. Also, estimate the number of students scoring marks more than 40 but less than 45.

Marks	$30-40$	$40-50$	$50-60$	$60-70$	$70-80$
No. of Students	31	42	51	35	31

Ans: For <45 marks, no. of students= 48, and for 40-45 marks, number of students=17.

9. If the number of persons earning below 1000 is 6000, estimate the number of persons having incomes between 2000 and 2500 from the following data:

Income	$1000-2000$	$2000-3000$	$3000-4000$	$4000-5000$
No. of Persons	4250	3600	1500	650

Ans: For income<2500, no. of persons= 12286 and for income 2000-2500, no. of persons=2036.

10. A survey conducted in a slum locality reveals the following information as classified below.

Income per day(Rs)	*under* 10	$10-20$	$20-30$	$30-40$	$40-50$
No. of Students	20	45	115	210	115

Estimate the probable number of persons in the income group 20 to 25.

Ans: For <25, No. of students= 108 and for 20-25, no. of students=43.

11. Compute $u_{14.2}$ from the following table by applying Newton's backward interpolation formula.

x	10	12	14	16	18
u_x	0.24	0.281	0.381	0.352	0.384

Ans: $u_{14.2}$ is 0.3837

12. From the following data estimate the numbers of students who have scored less than 70 marks.

Marks	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100
No. of Students	41	62	65	50	17

Ans: 196

13. Given $\sin 45^0 = 0.7071$, $\sin 50^0 = 0.7660$, $\sin 55^0 = 0.8192$, $\sin 60^0 = 0.8660$, find $\sin 57^0$ using an appropriate interpolation formula.

Ans: sin 57^0 is 0.8387

14. Use Newton's forward interpolation formula to find y_{35} given $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, and $y_{50} = 243$. **Ans: y_{35} =394.375**

15. Find the interpolating polynomial by using Newton's backward interpolation formula given that $f(0) = 7.4720$, $f(1) = 7.5854$, $f(2) = 7.6922$, $f(3) = 7.8119$, $f(4) = 7.9252$. Hence determine $f(2.5)$ **Ans: $f(2.5)$ is 7.7507**

16. Find the interpolating polynomial satisfying $f(0) = 0$, $f(2) = 4$, $f(4) = 56$, $f(6) = 204$, $f(8) = 496$, $f(10) = 980$, and hence determine $f(3)$, $f(5)$ and $f(7)$ using Newton's forward interpolation formula. **Ans: f(x)= x^3-2x and f(3)=21, f(5)=115, f(7)=329.**

17. Find the interpolating polynomial satisfying $f(0) = 1$, $f(1) = 2$, $f(2) = 4$, $f(3) = 7$, $f(4) = 11$, $f(5) = 16$, $f(6) = 22$, $f(7) = 29$. Hence determine $f(1.5)$.

Ans: f(x)= $x^2/2+x/2+1$ and f(1.5)=2.875

Interpolation for unequal/equal intervals

- **Lagrange's Interpolation:**

If $y = f(x)$ takes the values $y_0, y_1, ..., y_n$ corresponding to $x = x_0, x_1, ..., x_n$, then the Lagrange's interpolation formula is as follows:

$$y = \frac{(x-x_1)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)} y_1 + ... + \frac{(x-x_0)(x-x_1)...(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})} y_n.$$

Problems on Lagrange's Interpolation:

1. Use Lagrange's interpolation formula to find y at $x = 10$ given

x	5	6	9	11
y	12	13	14	16

Ans: 14.6667

2. The following table gives the viscosity of an oil as a function of temperature. Use Lagrange's formula to find viscosity of oil at a temperature of 140°.

Temp	110	130	160	190
Viscosity	10.8	8.1	5.5	4.8

Ans: at (140°) viscosity= 7.03331

3. The following are the measurements T made on a curve recorded by oscillograph representing a change of current I due to a change in the conditions of an electric current.

T	1.2	2.0	2.5	3.0
I	1.36	0.58	0.34	0.20

Using Lagrange's formula, find I at $T = 1.6$ **Ans: at T=1.6, I=0.8946**

4. Find the distance moved by a particle and its acceleration at the end of 4 seconds, if the time verses velocity data is as follows:

t	0	1	3	4
v	21	15	12	10

Ans: d=54.9, a=-3.4

5. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data:

Year	1997	1999	2001	2002
Profit in Lakh of Rupees	43	65	159	248

Ans: 100

6. Find the distance moved by a particle and its acceleration at the end of 4 seconds, if the time verses velocity data is as follows:

t	0	1	3
v	12	-6	24

Ans: d=50.667 and a=59.

7. Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$, find by using Lagrange's formula, the value of $\log_{10} 656$. **Ans:** $\log_{10} 656 = 2.8168$
8. If $y(1) = -3,\ y(3) = 9,\ y(4) = 30,\ y(6) = 132,$ find the Lagrange's interpolation polynomial that takes the same values as y at the given points. **Ans:** $y = x^3 - 3x^2 + 5x - 6$
9. Certain corresponding values of x and $\log_{10} x$ are: (300, 2.4771), (304, 2.4829), (305, 2.4843) and (307, 2.4871). Find $\log_{10} 301$ by using Lagrange's interpolation formula.

Ans: $\log_{10} 301 = 2.4786$

10. A curve passes through the point (0, 18), (1, 10), (3, -18) and (6, 90). Find the slope of the curve at $x = 2$. Ans: -16.
11. Apply appropriate Lagrange formula to find a root of the equation f(x)=0 given that f(30)= -30, f(34)= -13, f(38)= 3, f(42)=18. Ans: x=37.23

➢ **Lagrange's Inverse Interpolation:**

Lagrange's formula is a relation between two variables either of which may be taken as the independent variable. Therefore, on interchanging x and y in the Lagrange's formula, we obtain

$$x = \frac{(y-y_1)(y-y_2)...(y-y_n)}{(y_0-y_1)(y_0-y_2)...(y_0-y_n)} x_0 + \frac{(y-y_0)(y-y_2)...(y-y_n)}{(y_1-y_0)(y_1-y_2)...(y_1-y_n)} x_1 + ... + \frac{(y-y_0)(y-y_1)...(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)...(y_n-y_{n-1})} x_n$$

Problems on inverse Lagrange's Interpolation:

1. Applying Lagrange's formula inversely find x when $y = 6$ given the data

x	20	30	40
y	2	4.4	7.9

Ans: 32.246

2. Compute the value of x, when y = 8 by inverse interpolation using Lagrange's formula

x	-2	-1	1	2
y	-7	2	0	11

Ans: -3.421

3. Apply Lagrange's interpolation formula to obtain a root of the equation $f(x) = 0$, given that $f(30) = -30,\ f(34) = -13,\ f(38) = 3$ and $f(42) = 18$. Ans: 37.2304

4. Apply Lagrange's interpolation to find the value of x when $f(x)=15$ from the given data:

x	5	6	9	11
$f(x)$	12	13	14	16

Ans: 11.5

5. Obtain the value of t when $A=85$ from the following table using Lagrange's interpolation.

t	2	5	8	14
A	94.8	87.9	81.3	68.7

Ans: 6.3038

6. Use Lagrange's inverse interpolation formula to find the value of x for y = 100 given y(3) = 6, y(5) = 24, y(7) = 58, y(9) = 108, y(11) = 174. Ans: 8.6557

Numerical Integration

$$I=\int_a^b y\,dx=\int_{x_0}^{x_0+nh} f(x)dx$$

where $f(x)$ takes the values $y_0, y_1,..., y_n$ corresponding to $x=x_0, x_1,..., x_n$, then

- **Simpson's $\frac{1}{3}^{rd}$ Rule (n is multiple of 2)**

$$\int_{x_0}^{x_0+nh} f(x)dx=\frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\ ...\ +y_{n-2})+4(y_1+y_3\ +...+y_{n-1})\right].$$

- **Simpson's $\frac{3}{8}^{th}$ Rule (n is multiple of 3)**

$$\int_{x_0}^{x_0+nh} f(x)dx=\frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+\ ...\ +y_{n-1})+2(y_3+y_6\ +...+y_{n-3})\right].$$

- **Weddle's Rule (n is multiple of 6)**

$$\int_{x_0}^{x_6} f(x)dx=\frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right].$$

  - When $n=12$

$$\int_{x_0}^{x_0+nh} f(x)dx=\frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+2y_6+5y_7+y_8+6y_9+y_{10}+5y_{11}+y_{12}\right]$$

Problems on Numerical Integration:

1. Find the approximate value of $\int_0^{\pi/2}\sqrt{\cos\theta}\,d\theta$ by Simpson's 1/3rd rule by dividing $\left[0,\frac{\pi}{2}\right]$ into 6 equal parts. **Ans:** 68.023
2. Apply Simpson's 1/3rd rule with seven ordinates to evaluate $\int_2^8 \frac{dx}{\log_{10} x}$ **Ans:** 9.3848
3. Apply Simpson's 1/3rd rule to find $\int_0^{0.6} e^{-x^2}dx$ by taking 6 sub-intervals. **Ans:** 0.5351
4. Apply Simpson's 3/8th rule to evaluate $\int_1^4 e^{1/x}dx$. **Ans:** 4.9257

5. By using Simpson's 3/8th rule, evaluate $\int_0^{\pi/2} e^{\sin\theta} d\theta$. **Ans:** 177.8673

6. By using Simpson's 3/8th rule with h= 0.2 find the approximate area under the curve $y = \frac{x^2 - 1}{x^2 + 1}$ between the ordinates x = 1 and x = 2.8. Compare the result with the exact result.
Ans: 0.9152**,** Exact: 0.9152

7. Evaluate $\int_0^1 \frac{dx}{1+x}$ taking seven ordinates by applying Simpson's 3/8th rule. Hence deduce the value of $\log_e 2$. **Ans:** 0.6932; $\log_e 2 = 0.6932$

8. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using
(i) Simpson's 1/3rd rule taking four equal strips
(ii) Simpson's 3/8th rule taking six equal strips
(iii) Weddle's rule taking six equal strips
Hence compute an approximate value of π in each case. **Ans:** 0.7854, $\pi = 3.14156$

9. Evaluate $\int_0^1 \frac{x dx}{1+x^2}$ by Simpson's $1/3$ rule and $3/8^{th}$ rule taking seven ordinates and hence find $\log_e 2$. **Ans:** 0.3287 using $3/8^{th}$ rule

10. Find the value of $\int_2^6 \frac{dx}{x}$ by Simpson's $1/3$ rule (or $3/8^{th}$), Hence obtain approximate value of $\log_e 2$. **Ans:** 0.693

11. Given that

x	4	4.2	4.4	4.6	4.8	5	5.2
$y = \log_e x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $I = \int_4^{5.2} \log x dx$ by a) Simpson's 1/3rd rule, b) Simpson's 3/8th rule.
Also compare it with the exact value. **Ans**: a)1.8278472, b) 1.8278, Exact: 1.827822556

12. Find the distance travelled by a train between 8.20 AM and 9 AM from the following data.

Time	8.20 am	8.30 am	8.40 am	8.50 am	9 am
Speed (miles/hour)	24.2	35	41.3	42.8	39.4

Ans: 25.411 miles

13. Apply Simpson's rule to compute the area bounded by the curve y = f(x), x – axis and the extreme ordinates from the following table.

x	0	1	2	3	4	5	6
y	0	2	2.5	2.3	2	1.7	1.5

Ans:11.5125 sq.units

14. A river is 80 feet wide. The depth d in feet at a distance 'x' foot from one bank is given by

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Find approximately the area of cross section of the river. **Ans:** 710 sq. feet

15. The velocity 'v' of a particle at distance 's' from a point on its path is given by the table:

$s(ft)$	0	10	20	30	40	50	60
$v(fts^{-1})$	47	58	64	65	61	52	38

Estimate the time taken to travel $60\,ft$ by using Simpson's $1/3$ rd rule. **Ans**: 1.0635s

16. The velocity v(km/min) of a moped which starts from rest, is given at fixed intervals of time t(min) as follows:

t	2	4	6	8	10	12	14	16	18	20
v	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes. **Ans**: 292 km

17. A curve is drawn to pass through the points given by the following table:

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Apply Simpson's $1/3$ rule (or $3/8^{th}$) estimate the area bounded by the curve, the x-axis and the lines $x=1,\ x=4$. **Ans:** 7.8375

18. The following table gives the velocity v of a particle at time t:

$t(\sec)$	0	2	4	6	8	10	12
$v(m/\sec)$	4	6	16	34	60	94	136

Find the distance moved by the particle in 12 seconds.

Ans: 552 m.

19. Given that

x	4	4.2	4.4	4.6	4.8	5	5.2
$y=\log_e x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $I=\int_4^{5.2}\log x dx$ by Weddle's rule. **Ans:** 1.8278

20. A curve is drawn to pass through the points given by the following table:

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Apply Weddle's rule, estimate the area bounded by the curve, the x-axis and the lines $x=1,\ x=4$
. **Ans:**7.74 sq.units

21. A river is 80 feet wide. The depth d in feet at a distance x feet from one bank is given by

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Find approximately the area of cross section of the river. **Ans:** 710 sq.feet

22. A plane area is bounded by a curve, the x – axis and two extreme ordinates. The area is divided into six figures by equidistant ordinates 2 inches apart, the heights of the ordinates being 21.65, 21.04, 20.35, 19.61, 18.75, 17.80 and 16.75 respectively. Find the approximate value of the areas by numerical integration. ` **Ans:** 233.616 sq. inches