B.M.S. COLLEGE OF ENGINEERING, BENGALURU – 560 0 19
Autonomous college, affiliated to VTU

DEPARTMENT OF MATHEMATICS

Course: *Mathematical Foundation for Computer Science Stream-2(23MA2BSMCS)*

UNIT-2: VECTOR CALCULUS

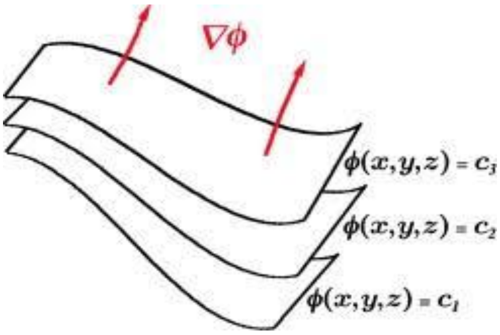
Scalar point function:

A Function $f:\mathbb{R}^3 \to \mathbb{R}$, whose values are scalars that depends on a point $P = P(x, y, z)$ i.e. $f = f(x, y, z)$.

Gradient of the scalar point function

If $f(x, y, z)$ is a scalar point function then the gradient of f is given by

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k.$$



Geometrically, $grad(f)$ is a normal to the surface $f(x, y, z) = c$ and has a magnitude equal to the rate of change of $f(x, y, z) = c$ along this normal.

1. Find $grad\,\phi$, if $\phi = \log(x^2 + y^2 + z^2)$. **Ans:** $\dfrac{2(x\,\hat{i} + y\,j + z\,k)}{x^2 + y^2 + z^2}$
2. If $f(x,y,z) = 3x^2y - y^3z^2$, find ∇f and $|\nabla f|$ at $(1,-2,-1)$.
   Ans: $\nabla f = -12\hat{\imath} - 9\hat{\jmath} - 16\hat{k}; |\nabla f| = \sqrt{481}$
3. If $f = x^2yz$ and $g = xy - 3z^2$, calculate $\nabla(\nabla f \cdot \nabla g)$.
   Ans: $(2y^2z + 3x^2z - 12xyz)\hat{\imath} + (4xyz - 6x^2z)\hat{\jmath} + (2xy^2 + x^3 - 6x^2y)\hat{k}$
4. The force in an electrostatic field given by $f = 4x^2 + 9y^2 + z^2$ has the direction of the gradient ∇f. Find this gradient at $P(5,-1,-11)$. **Ans:** $\nabla f = 40\hat{\imath} - \widehat{18}\jmath - 22\hat{k}$

Unit vector normal to the given surface:

Unit vector normal to the given surface $f(x, y, z)$ is given by $\hat{n} = \dfrac{\nabla f}{|\nabla f|}$

Find the unit vector normal to:

(i) the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$. **Ans:** $\dfrac{-\hat{i} + 3j + 2k}{\sqrt{14}}$

(ii) the cone of revolution $z^2 = 4(x^2 + y^2)$ at the point at the point $(1, 0, 2)$.

(iii) the surface $x^2y + 2zx = 4$ at the point $(2, -2, 3)$. **Ans:** $\frac{-\hat{\imath}+2\hat{\jmath}+2\hat{k}}{3}$

(iv) the surface $xy^3z^2 = 4$ at the point $(-1, -1, 2)$. **Ans:** $\frac{-\hat{\imath}-3\hat{\jmath}+\hat{k}}{\sqrt{11}}$

(v) the surface $x^2y - 2zx + 2y^2z^4 = 10$ at the point $(2,1,-1)$. **Ans:** $\frac{3\hat{\imath}+4\hat{\jmath}-6\hat{k}}{\sqrt{61}}$

Angle between two surfaces:

Angle between two surfaces f and g is given by $cos\theta = \frac{\nabla f.\nabla g}{|\nabla f||\nabla g|}$

1. Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at $(2,-1,2)$.
   Ans: $\cos^{-1}\frac{8\sqrt{21}}{63}$.
2. Calculate the angle between the normals to the surface $xy=z^2$ at the points $(4,1,2)$ and $(3,3,-3)$. **Ans:** $\cos^{-1}\left(-\frac{1}{\sqrt{11}}\right)$.
3. Show that the surfaces $4x^2y + z^4 = 12$ and $6x^2 - yz = 9x$ intersect orthogonally at the point (1, -1, 2)
4. Find the constants a and b so that the surfaces $ax^2-byz=(a+2)x$ is orthogonal to the surface $4x^2y+z^3=4$ at the point $(1,-1,2)$. **Ans:** $a=2.5$, $b=1$

Directional derivative

Directional derivative of ϕ in the direction of $\vec{a}$ at a point P is given by $\nabla\phi|_P \cdot a = \nabla\phi|_P \cdot \frac{\vec{a}}{|\vec{a}|}$.

Geometrically, the directional derivative of ϕ in the direction of a at a point P is the rate of change of ϕ at a point P in the direction of $\vec{a}$.

1. Find the directional derivatives of $f(x,y,z)=xy^2+yz^3$ at the point $(2,-1,1)$ in the direction of:
   a. the vector $\hat{i}+2j+2k$. **Ans:** $\frac{-11}{3}$
   b. the normal to the surface $x\log z-y^2=-4$ at $(-1,2,1)$. **Ans:** $\frac{15}{\sqrt{17}}$
2. Find the directional derivative of $f(x,y,z)=4e^{2x-y+z}$ at the point $(1,1,-1)$ in the direction towards the point $(-3,5,6)$.
3. Find the directional derivative of $x^2y^2z^2$ at the point $(1,1,-1)$ in the direction of the tangent to the curve $x=e^t$, $y=1+2\sin t$ and $z=t-\cos t$, where $-1\le t\le 1$. **Ans:** $\frac{4}{\sqrt{6}}$
4. The temperature of points in space is given $T(x,y,z)=x^2+y^2-z$. A mosquito located at $(1,1,2)$ desires to fly in such a direction it will get warm as soon as possible. In what direction should it move? **Ans:** $2i+2j-k$
5. The experiments show that the heat flows in the direction of maximum decrease of temperature T. Find this direction when the temperature $T=x^2+y^2+4z^2$ at the point $(2,-1,2)$.

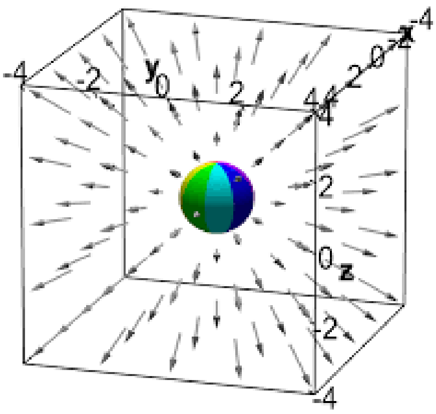
6. In which direction from $(3,1,-2)$ is the directional derivative of $\phi(x,y,z)=x^2y^2z^4$ maximum? Also find the magnitude of this maximum. **Ans:** $96(\hat{\imath}+3\hat{\jmath}-3\hat{k}); 96\sqrt{19}$
7. Find the values of the constants a, b and c so that the directional derivative of $f=axy^2+byz+cz^3x^3$ at $(1,2,-1)$ has a maximum of magnitude 64 in a direction parallel to the z-axis. **Ans:** $a=\frac{16}{3}, b=\frac{64}{3}, c=\frac{64}{9}$
8. If the directional derivative of $\phi=axy^2+byz+cz^2x^3$ at $(-1,1,2)$ has maximum magnitude of 32 units in the direction parallel to y-axis find a,b,c. **Ans:** $a=-12, b=4, c=1$

Vector point function

A Function $f:\mathbb{R}^3\to\mathbb{R}^3$, whose values are vectors that depends on a point $P=P(x,y,z)$ i.e. $\vec{f}(x,y,z)=f_1(x,y,z)\,\hat{i}+f_2(x,y,z)\,j+f_3(x,y,z)\,k$.

Divergence of a vector point function

The divergence of a continuously differentiable vector point function $\vec{f}=f_1\hat{i}+f_2 j+f_3 k$ is denoted by $div\,\vec{f}$ and is defined as $div\,\vec{f}=\nabla\cdot\vec{f}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$.



Physical interpretation: If $\vec{v}$ is the velocity field then $div(\vec{v})$ gives the rate at which fluid is originating at a point per unit volume. Alternately the divergence measures the outflow minus the inflow.

Solenoidal vector: A vector $\vec{F}=f_1\hat{i}+f_2 j+f_3 k$ is said to be solenoidal if $div\,\vec{F}=0$. (the flux entering any element of space is the same as that leaving it.)

Curl of a vector point function

The curl of a continuously differentiable vector point function $\vec{f}=f_1\hat{i}+f_2 j+f_3 k$ is denoted by $curl\,\vec{f}$ and is defined as:

$$curl\,\vec{f}=\nabla\times\vec{f}=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\times f_1\hat{i}+f_2 j+f_3 k=\begin{vmatrix}\hat{i} & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}.$$

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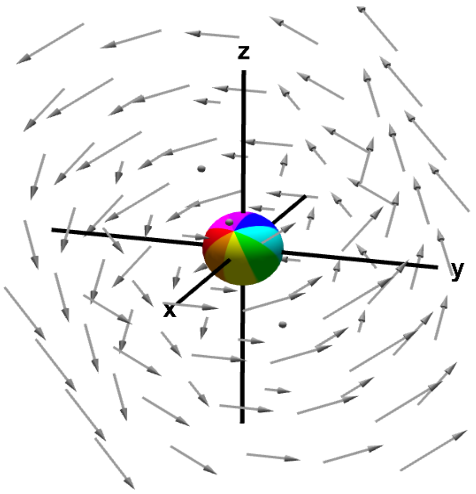

Physical interpretation: If $\vec{v}$ is the linear velocity and $\vec{\omega}$ is the angular velocity, then $\vec{v} = \vec{\omega} \times \vec{r}$ and $\nabla \times \vec{v} = 2\vec{\omega}$. Hence $curl\,\vec{F}$ gives the measure of the rotation of the vector field at any point.

Irrotational vector: A vector $\vec{F} = f_1\hat{i} + f_2 j + f_3 k$ is said to be irrotational if $curl\,\vec{F} = 0$. (any motion in which the angular velocity at any point is zero.)

1. Evaluate
   a. $div\left[3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k}\right]$ at the point $(1,2,3)$.
   b. $\nabla.\left(2x^2z\hat{i} - xy^2z\hat{j} + 3yz^2\hat{k}\right)$ at the point $(1,1,1)$.
   c. $\nabla.\left(e^y \sin x \cos z\ \hat{i} + e^{-x} \sin y \cos z\ \hat{j} + z^2 e^z\ \hat{k}\right)$.
   d. $\nabla.\left(3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}\right)$.
2. Show that each of following vectors are solenoidal:
   (i) $\left(-x^2 + yz\right)\hat{i} + \left(4y - z^2x\right)\hat{j} + \left(2xz - 4z\right)\hat{k}$.
   (ii) $3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$.
   (iii) $(x+3y)\hat{i} + (y-3z)j + (x-2z)k$
   (iv) $3y^4z^2\hat{i} + 4x^3z^2 j + 3x^2y^2 k$
3. If $\vec{F} = (x+y+1)\hat{i} + j - (x+y)k$ show that $\vec{F}\cdot curl\,\vec{F} = 0$.
4. Find $div\,\vec{F}$ and $curl\,\vec{F}$, where $F = grad\left(x^3 + y^3 + z^3 - 3xyz\right)$.

   Ans: $div\,\vec{F} = 6(x+y+z),\ curl\,\vec{F} = 0$
5. Evaluate $curl\left(xyz\hat{i} + 3x^2y\,j + \left(xz^2 - y^2z\right)k\right)$ at the point $(1,2,3)$.

   Ans: $-2yz\hat{i} + \left(xy - z^2\right)j + x(6y - z)k$
   .
6. Find the value of a if the vector $\left(ax^2y + yz\right)\hat{i} + \left(xy^2 - xz^2\right)j + \left(2xyz - 2x^2y^2\right)k$ has zero divergences. Find the curl of the above vector which has zero divergence.

   Ans: $a = -2$, $4x(z - xy)\hat{i} + \left(y - 2yz + 4xy^2\right)j + \left(2x^2 + 2y^2 - z^2 - z\right)k$.
7. If $\vec{r} = x\hat{i} + y\,j + zk$ and $r = \left|\vec{r}\right|$, show that $r^n\,\vec{r}$ is solenoidal for $n = -3$ and irrotational for all n.

8. If $f=\left(x^2+y^2+z^2\right)^{-n}$, find $div\left(grad\ f\right)$ and determine n if $div\left(grad\ f\right)=0$.

9. Find f, given

   a. $\nabla f=2xyz^3\,\hat{i}+x^2z^3\,\hat{j}+3x^2yz^2\,\hat{k}$ if $f\left(1,-2,2\right)=4$.

   b. $\nabla f=\left(y^2-2xyz^3\right)\hat{i}+\left(3+2xy-x^2z^3\right)\hat{j}+\left(6z^3-3x^2yz^2\right)\hat{k}$ if $f\left(1,0,1\right)=8$.

   c. $\nabla f=2x\hat{i}+4y\hat{j}+8z\hat{k}$.

   d. $\nabla f=\frac{1}{z^2}\left(zy\hat{i}+xz\hat{j}-xy\hat{k}\right)$.

   e. $\nabla f=xy\hat{\imath}+x^2y\hat{\jmath}$

10. A vector field is given by $F=\left(x^2-y^2+x\right)\hat{i}-\left(2yx+y\right)j$. Show that the field is irrotational and find its scalar potential. **Ans:** $\phi=\frac{x^3}{3}-xy^2+\frac{x^2}{2}-\frac{y^2}{2}$.

11. A vector field is given by $F=\left(6xy+z^3\right)\hat{i}+\left(3x^2-z\right)j+\left(3xz^2-y\right)k$. Show that the field is irrotational and find its scalar potential.

12. A fluid motion is given by $V=\left(y+z\right)\hat{i}+\left(z+x\right)j+\left(x+y\right)k$. Is this motion irrotational? If so, find the velocity potential. **Ans:** Yes. $\phi=xy+yz+zx$.

13. If $\vec{F}=2xyz^2\,\hat{i}+\left(x^2z^2+z\cos yz\right)j+\left(2x^2yz+y\cos yz\right)k$, show that $\vec{F}$ is a potential field or conservative field and hence find its scalar potential

14. A vector field is given by $F=\left(x^2-y^2+x\right)\hat{i}-\left(2yx+y\right)j$. Show that the field is irrotational and find its scalar potential. **Ans:** $\phi=\frac{x^3}{3}-xy^2+\frac{x^2}{2}-\frac{y^2}{2}$.

15. A fluid motion is given by $V=\left(y+z\right)\hat{i}+\left(z+x\right)j+\left(x+y\right)k$. Is this motion irrotational? If so, find the velocity potential. **Ans:** Yes. $\phi=xy+yz+zx$.

16. If $\vec{r}=x\hat{i}+yj+zk$ and $r=\left|\vec{r}\right|$, prove that

    a. $div\left(grad\ r^n\right)=\nabla^2\left(r^n\right)=n\left(n+1\right)r^{n-2}$

    b. $\nabla^2 f\left(r\right)=f''\left(r\right)+\frac{2}{r}f'\left(r\right)$.

    c. $grad(div\ \hat{r})=\frac{-2r}{r^3}$

ORTHOGONAL CURVILINEAR CO-ORDINATES

1. Prove that the cylindrical polar coordinate system is orthogonal curvilinear coordinate system.
2. Prove that the spherical polar coordinate system is orthogonal curvilinear coordinate system.
3. Express the vector $\vec{F}=x\hat{\imath}+2y\hat{\jmath}+yz\hat{k}$ in spherical polar coordinates.
4. Express the vector $\vec{F}=-z\hat{\imath}-2x\hat{\jmath}+y\hat{k}$ in spherical polar coordinates.
5. Express the vector $\vec{F}=2x\hat{\imath}+3y\hat{\jmath}-z\hat{k}$ in cylindrical polar coordinates.
6. Express the vector $\vec{F}=2y\hat{\imath}+z\hat{\jmath}-3x\hat{k}$ in cylindrical polar coordinates.