DEPARTMENT OF MATHEMATICS

Course: *Mathematical Foundation for Mechanical and Civil Engineering Stream 2(23MA2BSMCM)*
Mathematical Foundation for Electrical Stream-2(23MA2BSMES)

UNIT-2: VECTOR CALCULUS

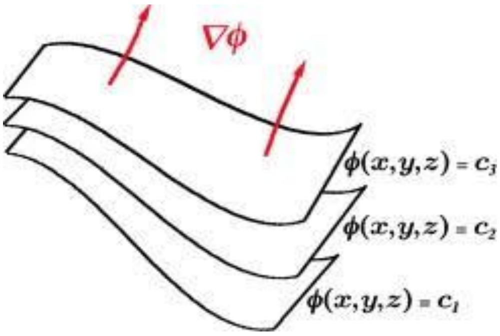
Scalar point function:

A Function $f:\mathbb{R}^3 \to \mathbb{R}$, whose values are scalars that depends on a point $P = P(x, y, z)$ i.e. $f = f(x, y, z)$.

Gradient of the scalar point function

If $f(x, y, z)$ is a scalar point function then the gradient of f is given by

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k .$$



Geometrically, $grad(f)$ is a normal to the surface $f(x, y, z) = c$ and has a magnitude equal to the rate of change of $f(x, y, z) = c$ along this normal.

1. Find $grad\,\phi$, if $\phi = \log(x^2 + y^2 + z^2)$. **Ans:** $\frac{2(x\,\hat{i} + y\,j + z\,k)}{x^2 + y^2 + z^2}$
2. If $f(x,y,z) = 3x^2y - y^3z^2$, find ∇f and $|\nabla f|$ at $(1,-2,-1)$.
   Ans: $\nabla f = -12\hat{\imath} - 9\hat{\jmath} - 16\hat{k}; |\nabla f| = \sqrt{481}$
3. If $f = x^2yz$ and $g = xy - 3z^2$, calculate $\nabla(\nabla f \cdot \nabla g)$.
   Ans: $(2y^2z + 3x^2z - 12xyz)\hat{\imath} + (4xyz - 6x^2z)\hat{\jmath} + (2xy^2 + x^3 - 6x^2y)\hat{k}$
4. The force in an electrostatic field given by $f = 4x^2 + 9y^2 + z^2$ has the direction of the gradient ∇f. Find this gradient at $P(5,-1,-11)$.

Unit vector normal to the given surface:

Unit vector normal to the given surface $f(x, y, z)$ is given by $\hat{n} = \frac{\nabla f}{|\nabla f|}$

Find the unit vector normal to:

(i) the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$. **Ans:** $\frac{-\hat{i} + 3j + 2k}{\sqrt{14}}$

(ii) the cone of revolution $z^2 = 4(x^2 + y^2)$ at the point at the point $(1, 0, 2)$.

(iii) the surface $x^2y + 2zx = 4$ at the point $(2, -2,3)$. **Ans:** $\frac{-\hat{\imath}+2\hat{\jmath}+2\hat{k}}{3}$

(iv) the surface $xy^3z^2 = 4$ at the point $(-1,-1,2)$. **Ans:** $\frac{-\hat{\imath}-3\hat{\jmath}+\hat{k}}{\sqrt{11}}$

(v) the surface $x^2y - 2zx + 2y^2z^4 = 10$ at the point $(2,1,-1)$. **Ans:** $\frac{3\hat{\imath}+4\hat{\jmath}-6\hat{k}}{\sqrt{61}}$

Angle between two surfaces:

Angle between two surfaces f and g is given by $cos\theta = \frac{\nabla f.\nabla g}{|\nabla f||\nabla g|}$

1. Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at $(2,-1,2)$.

Ans: $\cos^{-1}\frac{8\sqrt{21}}{63}$.

2. Calculate the angle between the normals to the surface $xy=z^2$ at the points $(4,1,2)$ and $(3,3,-3)$.

Ans: $\cos^{-1}\left(-\frac{1}{\sqrt{11}}\right)$.

3. Show that the surfaces $4x^2y + z^4 = 12$ and $6x^2 - yz = 9x$ intersect orthogonally at the point (1, -1, 2)

4. Find the constants a and b so that the surfaces $ax^2-byz=(a+2)x$ is orthogonal to the surface $4x^2y+z^3=4$ at the point $(1,-1,2)$. **Ans:** $a=2.5$, $b=1$

Directional derivative

Directional derivative of ϕ in the direction of $\vec{a}$ at a point P is given by $\nabla\phi|_P \cdot a = \nabla\phi|_P \cdot \frac{\vec{a}}{|\vec{a}|}$.

Geometrically, the directional derivative of ϕ in the direction of a at a point P is the rate of change of ϕ at a point P in the direction of $\vec{a}$.

1. Find the directional derivatives of $f(x,y,z)=xy^2+yz^3$ at the point $(2,-1,1)$ in the direction of:

   a. the vector $\hat{i}+2j+2k$. **Ans:** $\frac{-11}{3}$

   b. the normal to the surface $x\log z-y^2=-4$ at $(-1,2,1)$. **Ans:** $\frac{15}{\sqrt{17}}$

2. Find the directional derivative of $f(x,y,z)=4e^{2x-y+z}$ at the point $(1,1,-1)$ in the direction towards the point $(-3,5,6)$.

3. Find the directional derivative of $x^2y^2z^2$ at the point $(1,1,-1)$ in the direction of the tangent to the curve $x=e^t$, $y=1+2\sin t$ and $z=t-\cos t$, where $-1\le t\le 1$.

4. The temperature of points in space is given $T(x,y,z)=x^2+y^2-z$. A mosquito located at $(1,1,2)$ desires to fly in such a direction it will get warm as soon as possible. In what direction should it move? **Ans:** $2i+2j-k$

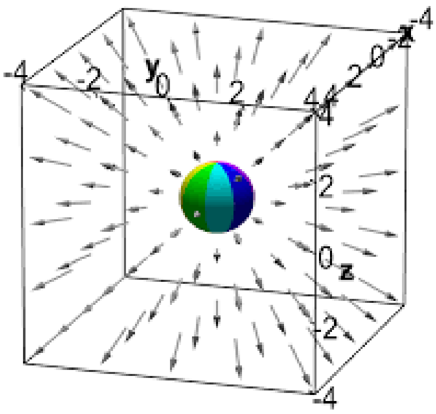
5. The experiments show that the heat flows in the direction of maximum decrease of temperature T. Find this direction when the temperature $T = x^2 + y^2 + 4z^2$ at the point $(2,-1,2)$.
6. In which direction from $(3,1,-2)$ is the directional derivative of $\phi(x,y,z) = x^2y^2z^4$ maximum? Also find the magnitude of this maximum. **Ans:** $96(\hat{\imath} + 3\hat{\jmath} - 3\hat{k}); 96\sqrt{19}$
7. Find the values of the constants a, b and c so that the directional derivative of $f = axy^2 + byz + cz^3x^3$ at $(1, 2, -1)$ has a maximum of magnitude 64 in a direction parallel to the z-axis. **Ans:** $a = \frac{16}{3}, b = \frac{64}{3}, c = \frac{64}{9}$
8. If the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(-1, 1, 2)$ has maximum magnitude of 32 units in the direction parallel to y-axis find a, b, c. **Ans:** $a = -12, b = 4, c = 1$

Vector point function

A Function $f:\mathbb{R}^3 \to \mathbb{R}^3$, whose values are vectors that depends on a point $P = P(x,y,z)$ i.e. $\vec{f}(x,y,z) = f_1(x,y,z)\,\hat{i} + f_2(x,y,z)\,j + f_3(x,y,z)\,k$.

Divergence of a vector point function

The divergence of a continuously differentiable vector point function $\vec{f} = f_1\hat{i} + f_2 j + f_3 k$ is denoted by $div\,\vec{f}$ and is defined as $div\,\vec{f} = \nabla\cdot\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$.



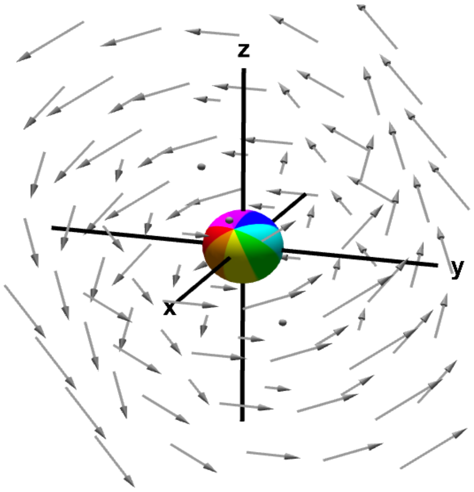
Physical interpretation: If $\vec{v}$ is the velocity field then $div(\vec{v})$ gives the rate at which fluid is originating at a point per unit volume. Alternately the divergence measures the outflow minus the inflow.

Solenoidal vector: A vector $\vec{F} = f_1\hat{i} + f_2 j + f_3 k$ is said to be solenoidal if $div\,\vec{F} = 0$. (the flux entering any element of space is the same as that leaving it.)

Curl of a vector point function

The curl of a continuously differentiable vector point function $\vec{f} = f_1\hat{i} + f_2 j + f_3 k$ is denoted by $curl\,\vec{f}$ and is defined as:

$$curl\,\vec{f} = \nabla\times\vec{f} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\times f_1\hat{i} + f_2 j + f_3 k = \begin{vmatrix} \hat{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}.$$



Physical interpretation: If $\vec{v}$ is the linear velocity and $\vec{\omega}$ is the angular velocity, then $\vec{v} = \vec{\omega} \times \vec{r}$ and $\nabla \times \vec{v} = 2\vec{\omega}$. Hence $curl\,\vec{F}$ gives the measure of the rotation of the vector field at any point.

Irrotational vector: A vector $\vec{F} = f_1\hat{i} + f_2 j + f_3 k$ is said to be irrotational if $curl\,\vec{F} = 0$. (any motion in which the angular velocity at any point is zero.)

1. Evaluate
   a. $div\left[3x^2\hat{i} + 5xy^2\hat{j} + xyz^3\hat{k}\right]$ at the point $(1,2,3)$.
   b. $\nabla.\left(2x^2z\hat{i} - xy^2z\hat{j} + 3yz^2\hat{k}\right)$ at the point $(1,1,1)$.
   c. $\nabla.\left(e^y \sin x \cos z\ \hat{i} + e^{-x} \sin y \cos z\ \hat{j} + z^2 e^z\ \hat{k}\right)$.
   d. $\nabla.\left(3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}\right)$.
2. Show that each of following vectors are solenoidal:
   (i) $\left(-x^2 + yz\right)\hat{i} + \left(4y - z^2x\right)\hat{j} + \left(2xz - 4z\right)\hat{k}$.
   (ii) $3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$.
   (iii) $(x+3y)\hat{i} + (y-3z)\,j + (x-2z)\,k$
   (iv) $3y^4z^2\hat{i} + 4x^3z^2\,j + 3x^2y^2\,k$
3. If $\vec{F} = (x+y+1)\hat{i} + j - (x+y)k$ show that $\vec{F} \cdot curl\,\vec{F} = 0$.
4. Find $div\,\vec{F}$ and $curl\,\vec{F}$, where $F = grad\left(x^3 + y^3 + z^3 - 3xyz\right)$.

   Ans: $div\,\vec{F} = 6(x+y+z)$, $curl\,\vec{F} = 0$
5. Evaluate $curl\left(xyz\hat{i} + 3x^2y\,j + \left(xz^2 - y^2z\right)k\right)$ at the point $(1,2,3)$.

   Ans: $-2yz\hat{i} + \left(xy - z^2\right)j + x(6y - z)k$.
6. Find the value of a if the vector $\left(ax^2y + yz\right)\hat{i} + \left(xy^2 - xz^2\right)j + \left(2xyz - 2x^2y^2\right)k$ has zero divergences. Find the curl of the above vector which has zero divergence.

   Ans: $a = -2$, $4x(z - xy)\hat{i} + \left(y - 2yz + 4xy^2\right)j + \left(2x^2 + 2y^2 - z^2 - z\right)k$.
7. If $\vec{r} = x\hat{i} + y\,j + zk$ and $r = |\vec{r}|$, show that $r^n\,\vec{r}$ is solenoidal for $n = -3$ and irrotational for all n.

8. If $f=\left(x^2+y^2+z^2\right)^{-n}$, find $div\left(grad\ f\right)$ and determine n if $div\left(grad\ f\right)=0$.

9. Find f, given
   a. $\nabla f=2xyz^3\,\hat{i}+x^2z^3\,\hat{j}+3x^2yz^2\,\hat{k}$ if $f(1,-2,2)=4$.
   b. $\nabla f=\left(y^2-2xyz^3\right)\hat{i}+\left(3+2xy-x^2z^3\right)\hat{j}+\left(6z^3-3x^2yz^2\right)\hat{k}$ if $f(1,0,1)=8$.
   c. $\nabla f=2x\hat{i}+4y\hat{j}+8z\hat{k}$.
   d. $\nabla f=\frac{1}{z^2}\left(zy\hat{i}+xz\hat{j}-xy\hat{k}\right)$.
   e. $\nabla f=xy\hat{\imath}+x^2y\hat{\jmath}$

10. A vector field is given by $F=\left(x^2-y^2+x\right)\hat{i}-\left(2yx+y\right)j$. Show that the field is irrotational and find its scalar potential. **Ans:** $\phi=\frac{x^3}{3}-xy^2+\frac{x^2}{2}-\frac{y^2}{2}$.

11. A vector field is given by $F=\left(6xy+z^3\right)\hat{i}+\left(3x^2-z\right)j+\left(3xz^2-y\right)k$. Show that the field is irrotational and find its scalar potential.

12. A fluid motion is given by $V=(y+z)\hat{i}+(z+x)j+(x+y)k$. Is this motion irrotational? If so, find the velocity potential. **Ans:** Yes. $\phi=xy+yz+zx$.

13. If $\vec{F}=2xyz^2\,\hat{i}+\left(x^2z^2+z\cos yz\right)j+\left(2x^2yz+y\cos yz\right)k$, show that $\vec{F}$ is a potential field or conservative field and hence find its scalar potential

14. A vector field is given by $F=\left(x^2-y^2+x\right)\hat{i}-\left(2yx+y\right)j$. Show that the field is irrotational and find its scalar potential. **Ans:** $\phi=\frac{x^3}{3}-xy^2+\frac{x^2}{2}-\frac{y^2}{2}$.

15. A fluid motion is given by $V=(y+z)\hat{i}+(z+x)j+(x+y)k$. Is this motion irrotational? If so, find the velocity potential. **Ans:** Yes. $\phi=xy+yz+zx$.

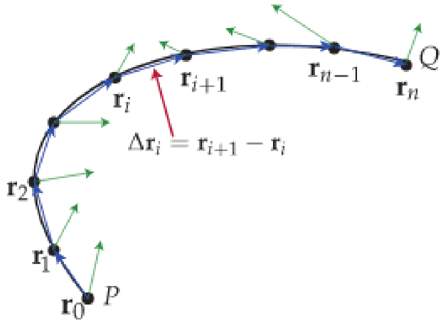
16. If $\vec{r}=x\hat{i}+yj+zk$ and $r=\left|\vec{r}\right|$, prove that
   a. $div\left(grad\ r^n\right)=\nabla^2\left(r^n\right)=n(n+1)r^{n-2}$
   b. $\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r)$.
   c. $grad(div\ \hat{r})=\frac{-2r}{r^3}$

Vector Integration

Line integrals

If $\vec{F}$ is a vector field with continuous components defined along a smooth curve C parametrized by $\vec{r}(t)$ with $a \le t \le b$, then the line integral of $\vec{F}$ along a curve C denoted by $\int_C \vec{F}\cdot d\vec{r}$ can be seen as

$$\int_C \vec{F}\cdot d\vec{r} = \lim_{n\to\infty}\left[\sum_{i=1}^{n} \vec{F}\left(\vec{r_i}\right)\cdot \Delta\vec{r_i}\right].$$



It is given by

$$\int_C \vec{F}\cdot d\vec{r} = \int_P^Q F_1 dx + F_2 dy + F_3 dz$$

where $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and P and Q are points on the curve C.

Geometrically if $\vec{F}$ is the force acting on an object then $\int_C \vec{F}\cdot d\vec{r}$ is the work done in moving the object along the curve C from P to Q.

If $\vec{F}$ is a continuous velocity field then $\int_C \vec{F}\cdot d\vec{r}$ is the flow along the curve C from P to Q. If curve C simple closed curve then the flow is said to be the **circulation** of $\vec{F}$ around C.

In general, the line integral in a domain depends on the curve C joining the two points P and Q. If the vector field is such that the line integral is independent of the path, then such vector fields are called **conservative** fields.

Problems:

1. If $\vec{F} = 3xy\hat{i} - y^2 j$ evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is the curve in the xy-plane $y = 2x^2$ from $(0,0)$ *to* $(1,2)$. **Ans:** $\frac{-7}{6}$

   .

2. If $\vec{f} = (2y+3)\hat{i} + (xz)\hat{j} + (yz - x)\hat{k}$, evaluate the $\int_C \vec{f}.\vec{dr}$, where C is the curve $x = 2t^2$, $y = t$ and $z = t^3$ from the point (0,0,0) to the point (2,1,1).

3. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz - y)j + zk$ along
   a. the straight line from $(0,0,0)$ *to* $(2,1,3)$. **Ans:** 16.
   b. the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ *to* $x = 2$. **Ans:**

4. A vector field is given by $\vec{F} = (\sin y)\hat{i} + x(1+\cos y)\, j$. Evaluate the line integral over a circular path given by $x^2 + y^2 = a^2,\ z = 0$. **Ans:** πa^2

5. Compute the line integral $\int_C y^2 dx - x^2 dy$ about the triangle whose vertices are $(1,0)$, $(0,1)$ and $(-1,0)$. **Ans:** $\frac{-2}{3}$

6. Evaluate $\oint_C \vec{f}.\vec{dr}$, $\vec{f} = [2z,\ x,\ -y]$ and C is $\vec{r} = [\cos t, \sin t, 2t]$ from (1, 0, 0) to (1, 0, 4π).

Green's theorem in the plane

If $M(x,y)$, $N(x,y)$, M_y and N_x be continuous in a region R of the xy-plane bounded by a simple closed curve C, then $\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$.

Problems:

1. Apply Green's theorem to evaluate $\int_C (xy + y^2)dx + x^2 dy$, *where* C is bounded by $y = x$ and $y = x^2$ **Ans:** $-\frac{1}{20}$

2. Apply Green's theorem to evaluate $\int_C (3x - 8y^2)dx + (4y - 6xy)dy$, *where* C is bounded by $x = 0, y = 0$ and $x + y = 1$ **Ans:** $\frac{5}{3}$

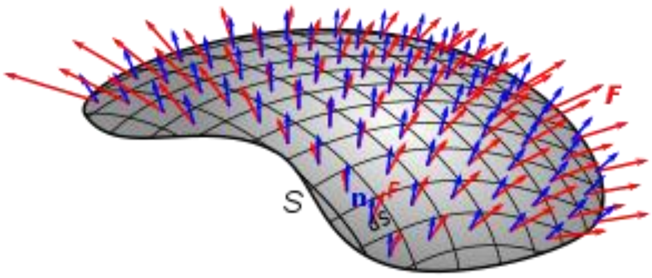
3. Apply Green's theorem, evaluate $\int_C (y - \sin x)dx + \cos x dy$ where C is the triangle enclosed by the lines $y = 0$, $x = \frac{\pi}{2}$ *and* $y = \frac{2x}{\pi}$. **Ans:** $-\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$

4. Using Green's theorem evaluate $\int_C (x^2 + y^2)\hat{\imath} - 2xy\,\hat{\jmath}$ where C is the rectangle bounded by $x = 0, x = 1, y = 0, y = 1$. **Ans:** -2

5. Apply Green's theorem to evaluate $\int_C (2x^2 - y^2)dx + (x^2 + y^2)dy$, *where* C is bounded by x −axis and upper half of the circle $x^2 + y^2 = a^2$ **Ans:** $\frac{4a^3}{3}$

6. Using Green's theorem evaluate $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$ where C is a square bounded by $x = -1, x = 1, y = -1, y = 1$. **Ans: 0**

Surface integrals



Orientable surface: A surface S is said to be orientable or two-sided surface if it is possible to define a field n of unit normal vectors on S that varies continuously with position. Once n has been chosen, we say that we have oriented the surface, and we call the surface together with its normal field an oriented surface. The vector n at that point is called the positive direction.

Example: Mobius strip is not an oriented surface.

The **surface integral** of a vector field $\vec{F}$ over an oriented or two-sided surface S is represented as

$$\int_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot n\, dS$$

where dS depends on the projection of the surface on the coordinate planes and is given

Projection of the surface	dS
On XY plane	$\dfrac{dxdy}{\lvert n\cdot k\rvert}$
On YZ plane	$\dfrac{dydz}{\lvert n\cdot \hat{i}\rvert}$
On XZ plane	$\dfrac{dxdz}{\lvert n\cdot j\rvert}$

Geometrically, the flux of a three-dimensional vector field $\vec{F}$ across an oriented surface S in the direction of n is given by the surface integral of the vector field $\vec{F}$.

Stoke's theorem

Let S be a piece-wise smooth oriented surface having a piecewise smooth boundary curve C and $\vec{F} = f_1\hat{i} + f_2\, j + f_3\, k$ be continuously differentiable vector point function on an open region containing S, then $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot n\, dS$ where n is a unit external normal vector at any point.

Problems:

1. Apply Stoke's theorem, evaluate $\oint_C (y+x)dx + (2x-z)dy + (y+z)dz$ where C is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ *and* $(0,0,6)$. **Ans:** 21.
2. c $\oint_C (2x-y)dx - yz^2dy - y^2zdz$ where C is the projection over the upper half of the sphere $x^2+y^2+z^2=a^2$ in the xy-plane. **Ans:** πa^2
3. If $f = 3yi - xzj + yz^2k$ and s is the surface of the paraboloid $2z = x^2+y^2$ bounded by z = 2, evaluate $\iint_S \left(\text{curl}\,\vec{f}\right)\cdot ndS$ using Stoke's theorem. **Ans:** -20π

4. Apply Stoke's theorem to evaluate $\oint_C ydx + xz^3dy - zy^3dz$ where C is the circle $x^2 + y^2 = 4$ in $z = 1.5$. **Ans:** $\frac{19\pi}{2}$

5. Evaluate $\int_C xy\,dx + xy^2dy$ using Stoke's theorem where C is the square in the x-y plane with vertices (1,0), (-1,0), (0,1) and (0,-1) respectively. **Ans:** $\frac{4}{3}$

6. If f=$(x^2 + y^2)i - 2xyj$ evaluate evaluate $\oint_C f.dr$ using Stoke's theorem, where C is is bounded by the lines $x = -a, x = a, y = 0, y = b$. **Ans**: $-4ab^2$

7. Apply Stoke's theore to evaluate $\oint_C \vec{F} \cdot d\vec{r}$, where $F = (x^2 - y^2)i + 2xyj$ over the rectangular box bounded by the planes $x = 0, x = a, y = 0, y = b, z = 0, z = c$ with the face $z = 0$ removed. **Ans**: $2ab^2$