

**Course:** Mathematical Foundation for Electrical Stream- 2 (23MA2BSMES)

Mathematical Foundation for Computer Science Stream- 2 (23MA2BSMCS)

Unit 1: INTEGRAL CALCULUS

Fubini's Theorem (First form)

If $f(x, y)$ is continuous on the rectangular region $R: a \leq x \leq b, c \leq y \leq d$, then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

Fubini's Theorem (Second form)

Let $f(x, y)$ be the continuous function on the rectangular region R .

I. If R is defined by $a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then $\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$

II. If R is defined by $c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then $\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$

The area of a closed, bounded region R in polar coordinate plane is $A = \iint_R r dr d\theta$ The volume of a closed, bounded region D in space is $V = \iiint_D dV$

DOUBLE INTEGRALS

I. Evaluate the following:

- $\int_0^1 \int_0^1 \frac{dx dy}{\sqrt{(1-x^2)(1-y^2)}}.$ Ans: $\frac{\pi^2}{4}.$
- $\int_1^2 \int_3^4 (xy + e^y) dy dx.$ Ans: $\frac{21+4(e^4-e^3)}{4}.$
- $\int_3^4 \int_1^2 \frac{dy dx}{(x+y)^2}.$ Ans: $\log\left(\frac{25}{24}\right).$
- $\int_1^2 \int_0^x \frac{dy dx}{x^2+y^2}.$ Ans: $\frac{\pi}{2} \log(2).$
- $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy dx}{1+x^2+y^2}$ Ans: $\frac{\pi}{4} \log(1 + \sqrt{2}).$
- $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dy dx.$ Ans: $\frac{3}{35}.$
- $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y} dx dy$ Ans: $\frac{\ln(8)^2}{2} - \ln(8) + \frac{1}{2}.$



II. Evaluate the following over the specified region:

1. $\iint_R xy dx dy$, where R is the region bounded by the circle $x^2 + y^2 = a^2$ in the first quadrant.
Ans: $\frac{a^4}{8}$
2. $\iint_A xy(x+y) dA$, where A is the area bounded by the parabola $y = x^2$ and the line $y = x$.
Ans: $\frac{3}{56}$
3. $\iint_R xy dx dy$, where R is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.
Ans: $\frac{a^4}{3}$
4. $\iint_D x^2 dx dy$, where D is the domain in the first quadrant bounded by the hyperbola $xy = 16$, and the lines $y = x$, $y = 0$ and $x = 8$.
Ans: 448

III. Change the order of integration and hence evaluate the following:

1. $\int_0^a \int_y^a \frac{x dx dy}{x^2 + y^2}$ Ans: $\frac{\pi a}{4}$.
2. $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dy dx}{\sqrt{y^4 - a^2 x^2}}$ Ans: $\frac{\pi a^2}{6}$.
3. $\int_0^1 \int_x^1 \frac{1}{1+y^4} dy dx$ Ans: $\frac{\pi}{8}$
4. $\int_0^1 \int_x^{\sqrt{x}} xy dy dx$ Ans: $\frac{1}{24}$.
5. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$ Ans: 1
6. $\int_0^\infty \int_0^x x e^{-x^2/y} dy dx$ Ans: $\frac{1}{2}$.
7. $\int_0^3 \int_0^{\sqrt{4-y}} (x+y) dx dy$ Ans: 601/60.
8. $\int_0^1 \int_{e^x}^e \frac{dy dx}{\log y}$ Ans: $e - 1$.
9. $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2) dx dy$ Ans: $\frac{a^3}{28} + \frac{a}{20}$.
10. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2} dy dx$ Ans: $\frac{\pi a^3}{6}$.
11. $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy dx$ Ans: $\frac{16}{3} a^2$.
12. $\int_0^1 \int_{x^2}^{2-x} xy dx dy$ Ans: $\frac{3}{8}$.
13. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy dx$ Ans: $1 - 1/\sqrt{2}$.

IV. Evaluate the following by transforming into polar coordinates:

1. $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2 + y^2) dy dx$ Ans: $\frac{\pi a^4}{8}$.



2. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$ Ans: $\frac{\pi}{4}$.
3. $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2} dy dx$ Ans: 9π .
4. $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2} dx dy$ Ans: $\frac{\pi}{4}(e-1)$.
5. $\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} \log_e(x^2+y^2) dx dy$, where $(a > 0)$ Ans: $\frac{\pi a^2}{4}(\log a - 1)$
6. $\int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}} dx dy$ Ans: $\frac{a^3}{3} \log(\sqrt{2} + 1)$

V. Area of the region using double integral

1. Find the area lying inside the cardioid $r = a(1 + \cos \theta)$ and outside the circle $r = a$.
Ans: $a^2(2 + \pi/4)$
2. Find the area of Lemniscate $r^2 = a^2 \cos 2\theta$. Ans: a^2
3. Find the area of the cardioid $r = a(1 + \cos \theta)$. Ans: $\frac{3\pi a^2}{2}$
4. Find the area which is inside the circle $r = 3a \cos \theta$ and outside the cardioid
 $r = a(1 + \cos \theta)$ Ans: πa^2
5. Find the area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$.
Ans: $a^2(1 - \frac{\pi}{4})$

TRIPLE INTEGRALS

VI. Evaluate the following:

1. $\int_0^1 \int_0^2 \int_1^2 x^2 y z dx dy dz$ Ans: $\frac{7}{3}$
2. $\int_{-c}^c \int_{-b}^b \int_{-a}^a (x^2 + y^2 + z^2) dz dy dx$ Ans: $\frac{8}{3} abc(a^2 + b^2 + c^2)$.
3. $\int_0^2 \int_1^2 \int_0^{yz} xyz dx dy dz$ Ans: 0
4. $\int_0^6 \int_0^{6-x} \int_1^{6-x-z} dy dz dx$ Ans: 18
5. $\int_0^4 \int_0^{\frac{1}{2}} \int_0^{x^2} \frac{1}{\sqrt{x^2-y^2}} dy dx dz$ Ans: 0.5113.
6. $\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} x dx dy dz$. Ans: $\frac{3\pi a^4}{64}$
7. $\int_0^1 \int_0^x \int_0^{x+y} (x+y+z) dz dy dx$ Ans: $\frac{7}{8}$.
8. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz dz dy dx$ Ans: $\frac{1}{48}$
9. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz dy dx}{\sqrt{1-x^2-y^2-z^2}}$ Ans: $\frac{\pi^2}{8}$



$$10. \int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z) dy dx dz$$

Ans: 0

$$11. \int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z dz dx dy$$

Ans: $\frac{1}{4}(e-2)$.

$$12. \int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy dx dz$$

Ans: 8π

VII. Evaluate the following over the region R bounded by the planes

$x = 0, y = 0, z = 0$ and $x + y + z = 1$.

$$1. \iiint_R (x+y+z) dx dy dz.$$

Ans: $1/8$

$$2. \iiint_R xyz dx dy dz$$

Ans: $\frac{1}{720}$.

$$3. \iiint_R \frac{dx dy dz}{(1+x+y+z)^3}$$

Ans: $\frac{1}{2}(\log 2 - \frac{5}{8})$

VIII. Volume of the region using triple integral

1. Find the volume of the tetrahedron bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the co-ordinate planes.

Ans: $\frac{abc}{6}$

2. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Ans: $\frac{4}{3}\pi abc$

3. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$.

Ans: $\frac{4\pi a^3}{3}$

BETA AND GAMMA FUNCTIONS

Beta function is defined as

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad n > 0.$$

Gamma function is defined as

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, \quad m, n > 0.$$

Properties:

$$1. \Gamma(n) = 2 \int_0^\infty e^{-x^2} x^{2n-1} dx, \quad n > 0.$$

$$2. \Gamma(n+1) = n\Gamma(n), n \in \mathbb{R}$$

$$3. \Gamma(n+1) = n!, n \notin \mathbb{Z}^-$$

$$4. \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin(n\pi)}, \quad 0 < n < 1.$$

$$5. \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$6. \beta(m, n) = \beta(n, m)$$

$$7. \beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$8. \beta(m, n) = 2 \int_0^{\pi/2} (\sin \theta)^{2m-1} (\cos \theta)^{2n-1} dx, \quad m, n > 0$$

IX. Express the following in terms of Gamma functions:

$$1. \int_0^\infty e^{-kx} x^{p-1} dx, k > 0$$

Ans: $\frac{\Gamma(p)}{k^p}$.



$$2. \int_0^{\infty} \frac{x^4}{4^x} dx.$$

$$Ans: \frac{3}{4(\log 2)^5}.$$

$$3. \int_0^{\infty} 3^{-4x^2} dx.$$

$$Ans: \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$$

$$4. \int_0^1 x^m [\log_e x]^n dx, \text{ where } n \text{ is an integer and } m > -1. \quad Ans: (-1)^n n! / (m+1)^{n+1}.$$

$$5. \int_0^1 x^{q-1} \left[\log_e \left(\frac{1}{x} \right) \right]^{p-1} dx \quad (p > 0, q > 0).$$

$$Ans: \frac{\Gamma(p)}{q^p}$$

$$6. \int_0^{\infty} e^{-t^2} t^{2n-1} dt.$$

$$Ans: \frac{1}{2} \Gamma(n)$$

X. Express the following in terms of Gamma functions:

$$1. \int_0^{\pi/2} \sqrt{\cot \theta} d\theta.$$

$$Ans: \frac{\pi}{\sqrt{2}}$$

$$2. \int_0^{\pi/2} (\sqrt{\tan \theta} + \sqrt{\sec \theta}) d\theta.$$

$$Ans: \frac{\pi}{\sqrt{2}} + \frac{\sqrt{\pi} \Gamma(\frac{1}{4})}{2\Gamma(3/4)}$$

$$3. \int_0^1 x^m (1-x^n)^p dx.$$

$$Ans: \frac{1}{n} \frac{\Gamma(\frac{m+1}{n}) \Gamma(p+1)}{\Gamma(\frac{m+1}{n} + p+1)}$$

$$4. \int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

$$Ans: \frac{\sqrt{\pi} \Gamma(\frac{1}{4})}{4\Gamma(\frac{3}{4})}$$

XI. Prove the following:

$$1. \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}, \text{ given } \int_0^{\infty} \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi} \text{ and hence deduce the value of } \int_0^{\infty} \frac{dy}{1+y^4}.$$

$$2. \int_0^{\infty} x e^{-x^8} dx \times \int_0^{\infty} x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}.$$

$$3. \int_0^{\pi/2} \sqrt{\sin \theta} d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}} = \pi.$$

$$4. \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}.$$

$$5. \frac{\beta(m+1, n)}{m} = \frac{\beta(m, n)}{m+n} = \frac{\beta(m, n+1)}{n}.$$