



Course: *Mathematical Foundation for Civil and Mechanical Engineering Stream- 2*
(23MA2BSMCM)

Unit 1: INTEGRAL CALCULUS

Fubini's Theorem (First form)

If $f(x, y)$ is continuous on the rectangular region $R: a \leq x \leq b, c \leq y \leq d$, then

$$\iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy = \int_a^b \int_c^d f(x, y) dy dx$$

Fubini's Theorem (Second form)

If $f(x, y)$ is continuous on the rectangular region R .

(i) If R is defined by $a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

(ii) If R is defined by $c \leq y \leq d, h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$$

The area of a closed, bounded region R in polar coordinate plane is $A = \iint_R r dr d\theta$

The volume of a closed, bounded region D in space is $V = \iiint_D dV$

Mass of the lamina corresponding to the region R with variable density $\rho(x, y)$ is given by $m = \iint_R \rho(x, y) dx dy$.

1. DOUBLE INTEGRALS

I. Evaluate the following: -

1. $\int_0^1 \int_0^1 \frac{dx dy}{\sqrt{(1-x^2)(1-y^2)}}$. Ans: $\frac{\pi^2}{4}$.
2. $\int_1^2 \int_3^4 (xy + e^y) dy dx$. Ans: $\frac{21+4(e^4-e^3)}{4}$.
3. $\int_3^4 \int_1^2 \frac{dy dx}{(x+y)^2}$. Ans: $\log\left(\frac{25}{24}\right)$.
4. $\int_1^2 \int_0^x \frac{dy dx}{x^2+y^2}$. Ans: $\frac{\pi}{2} \log(2)$.
5. $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy dx}{1+x^2+y^2}$. Ans: $\frac{\pi}{4} \log(1 + \sqrt{2})$.
6. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) dy dx$. Ans: $\frac{3}{35}$.



$$7. \int_1^{\ln 8} \int_0^{\ln y} e^{x+y} dx dy$$

$$\text{Ans: } \frac{\ln(8)^2}{2} - \ln(8) + \frac{1}{2}.$$

II. Evaluate the following over the specified region: -

1. $\iint_R xy dx dy$, where R is the region bounded by the circle $x^2 + y^2 = a^2$ in the first quadrant.

$$\text{Ans: } \frac{a^4}{8}$$

2. $\iint_A xy(x+y) dA$, where A is the area bounded by the parabola $y = x^2$ and the line $y = x$.

$$\text{Ans: } \frac{3}{56}$$

3. $\iint_R xy dx dy$, where R is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

$$\text{Ans: } \frac{a^4}{3}$$

4. $\iint_D x^2 dx dy$, where D is the domain in the first quadrant bounded by the hyperbola $xy = 16$, and the lines $y = x$, $y = 0$ and $x = 8$.

$$\text{Ans: } 448$$

III. Change the order of integration and hence evaluate the following: -

$$1. \int_0^a \int_y^a \frac{x dx dy}{x^2 + y^2}$$

$$\text{Ans: } \frac{\pi a}{4}.$$

$$2. \int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dy dx}{\sqrt{y^4 - a^2 x^2}}$$

$$\text{Ans: } \frac{\pi a^2}{6}.$$

$$3. \int_0^1 \int_x^1 \frac{1}{1+y^4} dy dx$$

$$\text{Ans: } \frac{\pi}{8}$$

$$4. \int_0^1 \int_x^{\sqrt{x}} xy dy dx$$

$$\text{Ans: } \frac{1}{24}.$$

$$5. \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$$

$$\text{Ans: } 1$$

$$6. \int_0^\infty \int_0^x x e^{-x^2/y} dy dx$$

$$\text{Ans: } \frac{1}{2}.$$

$$7. \int_0^3 \int_0^{\sqrt{4-y}} (x+y) dx dy$$

$$\text{Ans: } 601/60.$$

$$8. \int_0^1 \int_{e^x}^e \frac{dy dx}{\log y}$$

$$\text{Ans: } e - 1.$$

$$9. \int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2) dx dy$$

$$\text{Ans: } \frac{a^3}{28} + \frac{a}{20}.$$

$$10. \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2} dy dx$$

$$\text{Ans: } \frac{\pi a^3}{6}.$$

$$11. \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy dx$$

$$\text{Ans: } \frac{16}{3} a^2.$$

$$12. \int_0^1 \int_{x^2}^{2-x} xy dx dy$$

$$\text{Ans: } \frac{3}{8}.$$

$$13. \int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dy dx$$

$$\text{Ans: } 1 - 1/\sqrt{2}.$$



IV. Evaluate the following by transforming into polar coordinates: -

- $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2 + y^2) dy dx$ Ans: $\frac{\pi a^4}{8}$.
- $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$ Ans: $\frac{\pi}{4}$.
- $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2 + y^2} dy dx$ Ans: 9π .
- $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2} dx dy$ Ans: $\frac{\pi}{4}(e - 1)$.
- $\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} \log_e(x^2 + y^2) dx dy$, where $(a > 0)$ Ans: $\frac{\pi a^2}{4}(\log a - 1)$
- $\int_0^a \int_y^a \frac{x}{\sqrt{x^2+y^2}} dx dy$ Ans: $\frac{a^3}{3} \log(\sqrt{2} + 1)$

V. Area of the region using double integral

- Find the area lying inside the cardioid $r = a(1 + \cos \theta)$ and outside the circle $r = a$.

$$\text{Ans: } a^2(2 + \pi/4)$$

- Find the area of Lemniscate $r^2 = a^2 \cos 2\theta$.

$$\text{Ans: } a^2$$

- Find the area of the cardioid $r = a(1 + \cos \theta)$.

$$\text{Ans: } \frac{3\pi a^2}{2}$$

- Find the area which is inside the circle $r = 3a \cos \theta$ and outside the cardioid

$$r = a(1 + \cos \theta)$$

$$\text{Ans: } \pi a^2$$

- Find the area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$.

$$\text{Ans: } a^2(1 - \frac{\pi}{4})$$

2. TRIPLE INTEGRALS

VII. Evaluate the following: -

- $\int_0^1 \int_0^2 \int_1^2 x^2 y z dx dy dz$ Ans: $\frac{7}{3}$
- $\int_{-c}^c \int_{-b}^b \int_{-a}^a (x^2 + y^2 + z^2) dz dy dx$ Ans: $\frac{8}{3} abc(a^2 + b^2 + c^2)$.
- $\int_0^2 \int_1^z \int_0^{yz} xy z dx dy dz$ Ans: 0
- $\int_0^6 \int_0^{6-x} \int_1^{6-x-z} dy dz dx$ Ans: 18
- $\int_0^4 \int_0^{\frac{1}{2}} \int_0^{x^2} \frac{1}{\sqrt{x^2-y^2}} dy dx dz$ Ans: 0.5113.
- $\int_0^a \int_0^{\sqrt{a^2-z^2}} \int_0^{\sqrt{a^2-y^2-z^2}} x dx dy dz$ Ans: $\frac{3\pi a^4}{64}$
- $\int_0^1 \int_0^x \int_0^{x+y} (x + y + z) dz dy dx$ Ans: $\frac{7}{8}$.



8. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz dz dy dx$ Ans: $\frac{1}{48}$
9. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dz dy dx}{\sqrt{1-x^2-y^2-z^2}}$ Ans: $\frac{\pi^2}{8}$
10. $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z) dy dx dz$ Ans: 0
11. $\int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z dz dx dy$ Ans: $\frac{1}{4}(e-2)$.
12. $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dy dx dz$ Ans: 8π

VIII. Evaluate the following over the region R bounded by the planes

$x = 0, y = 0, z = 0$ and $x + y + z = 1$.

1. $\iiint_R (x+y+z) dx dy dz$. Ans: $1/8$
2. $\iiint_R xyz dx dy dz$ Ans: $\frac{1}{720}$.
3. $\iiint_R \frac{dx dy dz}{(1+x+y+z)^3}$ Ans: $\frac{1}{2}(\log 2 - \frac{5}{8})$.

IX. Volume of the region using triple integral

1. Find the volume of the tetrahedron bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the co-ordinate planes. Ans: $\frac{abc}{6}$
2. Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Ans: $\frac{4}{3}\pi abc$
3. Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$. Ans: $\frac{4\pi a^3}{3}$

3. BETA AND GAMMA FUNCTIONS

I. Express the following in terms of Gamma functions: -

1. $\int_0^\infty e^{-kx} x^{p-1} dx, k > 0$ Ans: $\frac{\Gamma(p)}{k^p}$.
2. $\int_0^\infty \frac{x^4}{4^x} dx$. Ans: $\frac{3}{4(\log 2)^5}$.
3. $\int_0^\infty 3^{-4x^2} dx$. Ans: $\frac{\sqrt{\pi}}{4\sqrt{\log 3}}$
4. $\int_0^1 x^m [\log_e x]^n dx$, where n is an integer and $m > -1$. Ans: $(-1)^n n! / (m+1)^{n+1}$.
5. $\int_0^1 x^{q-1} \left[\log_e \left(\frac{1}{x} \right) \right]^{p-1} dx$ ($p > 0, q > 0$). Ans: $\frac{\Gamma(p)}{q^p}$
6. $\int_0^\infty e^{-t^2} t^{2n-1} dt$. Ans: $\frac{1}{2} \Gamma(n)$

II. Express the following in terms of Gamma functions: -

1. $\int_0^{\pi/2} \sqrt{\cot \theta} d\theta$. Ans: $\frac{\pi}{\sqrt{2}}$
2. $\int_0^{\pi/2} (\sqrt{\tan \theta} + \sqrt{\sec \theta}) d\theta$. Ans: $\frac{\pi}{\sqrt{2}} + \frac{\sqrt{\pi} \Gamma(\frac{1}{4})}{2\Gamma(3/4)}$



$$3. \int_0^1 x^m (1-x^n)^p dx.$$

$$Ans: \frac{1}{n} \frac{\Gamma\left(\frac{m+1}{n}\right) \Gamma(p+1)}{\Gamma\left(\frac{m+1}{n} + p + 1\right)}$$

$$4. \int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

$$Ans: \frac{\sqrt{\pi} \Gamma\left(\frac{1}{4}\right)}{4 \Gamma\left(\frac{3}{4}\right)}$$

III. Prove the following: -

$$1. \Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n\pi}, \text{ given } \int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi} \text{ and hence deduce the value of } \int_0^\infty \frac{dy}{1+y^4}.$$

$$2. \int_0^\infty x e^{-x^8} dx \times \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}.$$

$$3. \int_0^{\pi/2} \sqrt{\sin \theta} d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}} = \pi.$$

$$4. \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}.$$

$$5. \frac{\beta(m+1, n)}{m} = \frac{\beta(m, n)}{m+n} = \frac{\beta(m, n+1)}{n}.$$