B.M.S. COLLEGE OF ENGINEERING, BENGALURU – 560 0 19
Autonomous college, affiliated to VTU
DEPARTMENT OF MATHEMATICS

Course Code: 23MA1BSMCS, 23MA1BSCEM
Course: Mathematical Foundation for Computer Science Stream – 1
Mathematical Foundation for Civil, Electrical and Mechanical Engg. Stream – 1

I. RANK OF MATRICES:

1. Find the rank of the following matrices by reducing them into echelon form.

a) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$.
Ans: $\rho=2$

b) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$.
Ans: $\rho=2$

c) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$.
Ans: $\rho=3$

d) $\begin{bmatrix} 1 & 4 & 5 \\ 2 & 6 & 8 \\ 3 & 7 & 22 \end{bmatrix}$.
Ans: $\rho=3$

e) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$
Ans: $\rho=2$

f) $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$.
Ans: $\rho=2$

g) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$.
Ans: $\rho=2$

h) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$.
Ans: $\rho=3$

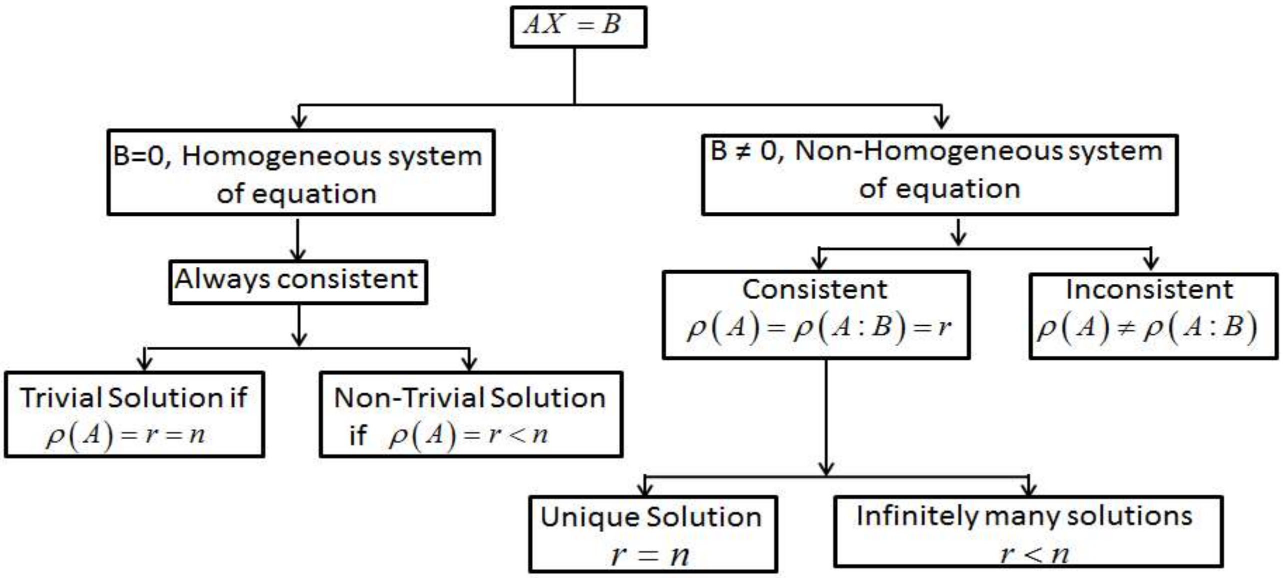
i) $\begin{bmatrix} 1 & -2 \\ 3 & -6 \\ 7 & -1 \\ 4 & 5 \end{bmatrix}$

j) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$.
Ans: $\rho=2$

k) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix}$.
Ans: $\rho=4$

2. Find 'b' if the rank of $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ b & 2 & 2 & 2 \\ 9 & 9 & b & 3 \end{bmatrix}$ is 3. Ans: $b=2$ or $b=-6$.

II. Consistency and solution of the linear system of equations:



$n =$ Number of Unknowns

1. Solve the following system of equations:

SL.NO	Questions	Answers
a	$x+2y+3z=0$ $3x+4y+4z=0$ $7x+10y+12z=0$	$x=y=z=0$
b	$2x_1+3x_2-4x_3+x_4=0$ $x_1-x_2+x_3+2x_4=0$ $5x_1-x_3+7x_4=0$ $7x_1+8x_2-11x_3+5x_4=0$	$x_1=\frac{(k_1-7k_2)}{5}, x_2=\frac{(6k_1+3k_2)}{5}, x_3=k_1, x_4=k_2$
c	$x_1-2x_2-x_3+3x_4=0$ $-2x_1+4x_2+5x_3-5x_4=0$ $3x_1-6x_2-6x_3+8x_4=0$	$x_1=\frac{(6k_1-10k_2)}{3}, x_2=k_1, x_3=\frac{-k_2}{2}, x_4=k_2$
d	$x+3y-2z=0$ $2x-y+4z=0$ $x-11y+14z=0$	$x=\frac{-10k}{7}, y=\frac{8k}{7}, z=k$
e	$4x+2y+z+3w=0$ $6x+3y+4z+7w=0$ $2x+y+w=0$	$x=\frac{(k_1+k_2)}{2}, y=k_1, z=-k_2, w=k_2$

2. Discuss the consistency of the following system of linear equations

a) $5x+3y+7z=4$, $3x+26y+2z=9$, $7x+2y+10z=5$

Ans: Consistent and has infinitely many solutions

$x=\frac{7-16k}{11}\quad y=\frac{k+3}{11}\quad z=k$

b) $4x-2y+6z=8$, $x+y-3z=-1$, $15x-3y+9z=21$

Ans: Consistent and has infinitely many solutions

$x=1,\ y=3k-2,\ z=k$

c) $2x-3y+7z=5$, $3x+y-3z=13$, $2x+19y-47z=32$

Ans: inconsistent

d) $2x_1+3x_2-x_3=1$, $3x_1-4x_2+3x_3=-1$, $2x_1-x_2+2x_3=-3$, $3x_1+1x_2-2x_3=4$

Ans: Inconsistent

e) $x+2y+z=3$, $2x+3y+2z=5$, $3x-5y+5z=2$, $3x+9y-z=4$

Ans: Consistent and has unique solution

$x=-1,\ y=1,\ z=2$

f) $2x+6y+11=0$, $6x+20y-6z+3=0$, $6y-18z+1=0$

Ans: inconsistent

g) $2x_1-2x_2+4x_3+3x_4=9$, $x_1-x_2+2x_3+2x_4=6$, $2x_1-2x_2+x_3+2x_4=3$, $x_1-x_2+x_4=2$

Ans: Inconsistent

h) $2x+y-z=0$, $2x+5y+7z=52$, $x+y+z=9$

Ans: Consistent and has unique solution

$x=1, y=3, z=5$

i) $3x+2y+2z=0$, $x+2y=4$, $10y+3z=-2$, $2x-3y-z=5$

Ans: Consistent and has unique solution

$x=2, y=1, z=-4$

j) $x_1+x_2-2x_3+x_4+3x_5=1$, $2x_1-x_2+2x_3+2x_4+6x_5=2$, $3x_1+2x_2-4x_3-3x_4-9x_5=3$

Ans: Consistent and has infinitely many solutions

$x_1=1,\ x_2=2a,\ x_3=a,\ x_4=-3b,\ x_5=b$

3. Investigate the values of λ and μ so that the equations $2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+\lambda z=\mu$ have

(i) no solution (ii) unique solution (iii) infinite number of solutions.

Ans: (i) If $\lambda=5$ and $\mu\neq 9$ (ii) $\lambda\neq 5$ and μ can be any value (iii) $\lambda=5$ and $\mu=9$.

4. Determine the values of a and b for which the system $x+2y+3z=6$, $x+3y+5z=9$, $2x+5y+az=b$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

Ans: (i) If $a=8$ and $b\neq 15$ (ii) $a\neq 8$ and for any b (iii) $a=8$ and $b=15$.

5. Find the value of a for which the system $x+2y+z=3$; $ay+5z=10$; $2x+7y+az=b$ has a unique solution. Also, find the pair of values (a,b) for which it has infinitely many solutions.

6. Find the values of a and b for which the system of equations $\begin{matrix} x+ay+z=3 \\ x+2y+2z=b \\ x+5y+3z=9 \end{matrix}$ is consistent.

Ans: If $a=-1$ and b=6 equations will be consistent and have an infinite number of solutions.
If $a\neq -1$ and b has any value, equations will be consistent and have a unique solution.

7. Show that the equations $\begin{matrix} 3x+4y+5z=a \\ 4x+5y+6z=b \\ 5x+6y+7z=c \end{matrix}$ do not have a solution unless $a+c=2b$.

8. Test for consistency $\begin{matrix} -2x+y+z=a \\ x-2y+z=b \\ x+y-2z=c \end{matrix}$ where a,b and c are constants.

Ans: if $a+b+c\neq 0$ inconsistent, if $a+b+c=0$, then infinitely many solutions.

III. Gauss elimination method:

1. Solve the following system of equations by Gauss elimination method:

SL NO	Questions	Answers
a	$2x_1+x_2+x_3=10$ $3x_1+2x_2+3x_3=18$ $x_1+4x_2+9x_3=16$	$x_1=7,\ x_2=-9,\ x_3=5.$
b	$2x+2y+z=12$ $3x+2y+2z=8$ $5x+10y-8z=10$	$x=\frac{-51}{4}, y=\frac{115}{8}, z=\frac{35}{4}$
c	$2x_1+4x_2+x_3=3$ $3x_1+2x_2-2x_3=-2$ $x_1-x_2+x_3=6$	$x_1=2,\ x_2=-1,\ x_3=3$
d	$10x+2y+z=9$ $2x+20y-2z=-44$ $-2x+3y+10z=22$	$x=1, y=-2, z=3$
e	$2x_1+x_2+4x_3=12$ $8x_1-3x_2+2x_3=20$ $4x_1+11x_2-x_3=33$	$x_1=3,\ x_2=2,\ x_3=1.$

SL NO	Questions	Answers
f	$x_1+4x_2-x_3=-5$ $x_1+x_2-6x_3=-12$ $3x_1-x_2-x_3=4$	$x_1=\frac{117}{71},\ x_2=-\frac{81}{71},$ $x_3=\frac{148}{71}$
g	$2x_1+x_2+3x_3=1$ $4x_1+4x_2+7x_3=1$ $2x_1+5x_2+9x_3=3$	$x_1=\frac{-1}{2},\ x_2=-1,\ x_3=1.$
h	$2x_1-7x_2+4x_3=9$ $x_1+9x_2-6x_3=1$ $-3x_1+8x_2+5x_3=6$	$x_1=4, x_2=1, x_3=2$
i	$5x_1+x_2+x_3+x_4=4$ $x_1+7x_2+x_3+x_4=12$ $x_1+x_2+6x_3+x_4=-5$ $x_1+x_2+x_3+4x_4=-6$	$x_1=1,\ x_2=2,$ $x_3=-1,\ x_4=-2$

2. Show that if $\lambda\neq -5$ the system of equations $\begin{matrix} 3x-y+4z=3 \\ x+2y-3z=-2 \\ 6x+5y+\lambda z=-3 \end{matrix}$ have a unique solution. If $\lambda=-5$ show that the equations are consistent. Determine the solution in each case.

Ans: when $\lambda\neq -5$, $x=\frac{4}{7},\ y=\frac{-9}{7},\ z=0$, when $\lambda=-5$, $x=\frac{4-5k}{7},\ y=\frac{13k-9}{7},\ z=k$.

3. Prove that the equations $\begin{matrix} 5x+3y+2z=12 \\ 2x+4y+5z=2 \\ 39x+43y+45z=c \end{matrix}$ are incompatible unless $c=74$; and in that case the equations are satisfied by $x=2+t, y=2-3t, z=-2+2t$, where t is any arbitrary quantity.

4. For what values of k, the equations $\begin{matrix} x+y+z=1 \\ 2x+y+4z=k \\ 4x+y+10z=k^2 \end{matrix}$ have a solution and solve them completely in each case.

   Ans: When $k=1,\ x=-3z, y=2z+1$, when $k=2, x=1-3z, y=2z.$

IV. Gauss-Seidel Iteration method:

If the system is diagonally dominant, then the method is known to give convergent solution.

Diagonal dominance is not a necessary condition.

Suppose $AX=B$ is diagonally dominant with $A=\left(a_{ij}\right),\ X=\left(x_i\right)$ and $B=\left(b_i\right)$.

Then the iterative formula is

$$x_1^{n+1}=\frac{1}{a_{11}}\left(b_1-a_{12}x_2^n-a_{13}x_3^n\cdots\cdots\right)$$
$$x_2^{n+1}=\frac{1}{a_{22}}\left(b_2-a_{21}x_1^{n+1}-a_{23}x_3^n\cdots\cdots\right)$$
$$x_3^{n+1}=\frac{1}{a_{33}}\left(b_3-a_{31}x_1^{n+1}-a_{32}x_2^{n+1}\cdots\cdots\right)$$
$$\vdots$$

Solve the following system of equations by Gauss-Seidel method.

SL.NO.	Questions	Answers
1	$20x+y-2z=17$ $3x+20y-z=-18$. $2x-3y+20z=25$	$x=1,\ y=-1,\ z=1$
2	$5x+2y+z=12$ $x+4y+2z=15$ $x+2y+5z=20$	$x=0.996,\ y=2,\ z=3$
3	$2x+y+6z=9$ $8x+3y+2z=13$ $x+5y+z=7$	$x=1,\ y=1,\ z=1$
4	$28x+4y-z=32$ $x+3y+10z=24$ $2x+17y+4z=35$	$x=0.9876,\ y=1.5090,\ z=1.8485$
5	$10x+2y+z=9$ $2x+20y-2z=-44$ $-2x+3y+10z=22$	$x=1,\ y=-2,\ z=3$

SL.NO.	Questions	Answers
6	$83x+11y-4z=95$ $7x+52y+13z=104$ $3x+8y+29z=71$	$x=1.06, y=1.37, z=1.96$
7	$54x+y+z=110$ $2x+15y+6z=72$ $-x+6y+27z=85$	$x=1.926,\ y=3.573,\ z=2.425$
8	$5x_1-x_2=9$ $-x_1+5x_2-x_3=4$ $-x_2+5x_3=-6$	$x_1=1.99,\ x_2=0.99,\ x_3=-1$
9	$8x_1+x_2-x_3=8$ $2x_1+x_2+9x_3=12$ $x_1-7x_2+2x_3=-4$	$x_1=1\ x_2=1,\ x_3=1$
10	$4x_1+2x_2+x_3=11$ $-x_1+2x_2=3$ $2x_1+x_2+4x_3=16$	$x_1=1,\ x_2=2, x_3=3$
11	Start with $(2,2,-1)$ and solve $5x_1-x_2+x_3=10$ $2x_1+4x_2=12$ $x_1+x_2+5x_3=-1$	$x_1=2.5555, x_2=1.7222,\ x_3=-1.0555$
12	$10x_1+x_2+x_3=12$ $2x_1+10x_2+x_3=13$ $2x_1+2x_2+10x_3=14$	$x_1=x_2=x_3=1$
13	$27x_1+6x_2-x_3=85$ $6x_1+15x_2+2x_3=72$ $x_1+x_2+54x_3=110$	$x_1=2.4255,\ x_2=3.573,\ x_3=1.926$
14	$x_1-8x_2+3x_3=-4$ $2x_1+x_2+9x_3=12$ $8x_1+2x_2-2x_3=8$	$x_1=x_2=x_3=1$
15	$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$	$x=1,\ y=2,\ z=3, u=4$

V. Characteristic values (Eigenvalues) and characteristic vectors (Eigenvectors):

If A is a square matrix, then λ is said to be an eigenvalue of the matrix if there exists a non-zero vector X such that $AX = \lambda X$. X is called the eigenvector corresponding to the eigenvalue λ.

$\lambda X = \lambda IX \Rightarrow (A-\lambda I)X = 0$. We seek non-trivial solutions of $(A-\lambda I)X = 0$.

X is non-trivial if $\rho(A-\lambda I) < n \Rightarrow |A-\lambda I| = 0$.

If A is a matrix of size 3×3 then $-\lambda^3 + Tr(A)\lambda^2 - \sum M_{ii}^{2\times 2}\lambda + |A| = 0$.

Find the eigenvalues and its corresponding eigenvectors for the following matrices:

a. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$. Ans: $\lambda = -2,3,6; x_1 = [-k,0,k], x_2 = [k,-k,k], x_3 = [k,2k,k]$

b. $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$, Ans: $\lambda = 2,3,5;\ x_1 = k_1[1,-1,0],\ x_2 = k_2[1,0,0],\ x_3 = k_3[3,2,1]$.

c. $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$, $\text{Ans}: \lambda = 0,3,15;\ x_1 = [1,2,2],\ x_2 = [2,1,-2],\ x_3 = [2,-2,1]$

d. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$, Ans: $\lambda = 1,2,3;\ x_1 = [1,0,-1], x_2 = [0,1,0], x_3 = [1,0,1]$

e. $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, $\text{Ans}: \lambda = 5,-3,-3;\ x_1 = k[1,2,-1],\ x_2 = [3k_1 - 2k_2, k_2, k_1]$

f. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$, Ans: $\lambda = 8,2,2;\ x_1 = [2,-1,1],\ x_2 = [1,0,-2],\ x_3 = [1,2,0]$

g. $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$, $\text{Ans}: \lambda = 1,-1,4;\ x_1 = [2,-1,1],\ x_2 = [0,1,1],\ x_3 = [-1,-1,1]$

h. $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$, $\text{Ans}: \lambda = 1,2,2;\ x_1 = k_1[1,1,-1],\ x_2 = k_2[2,1,0],\ x_3 = k_3[2,1,0]$

i. $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, Ans: $\lambda = 5,1,1;\ x_1 = [1,1,1],\ x_2 = [1,0,-1],\ x_3 = [2,-1,0]$

j. $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$, Ans: $\lambda = 2,2,3;\ x_1 = x_2 = [5,2,-3],\ x_3 = [1,1,-2]$

k. $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$, Ans: $\lambda = 2,2,-2;\ x_1 = x_2 = [0,1,1],\ x_3 = [-4,-1,1]$

l. $\begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix}$, Ans: $\lambda = -5, 2, 2;\ x_1 = [3,2,4],\ x_2 = x_3 = [1,3,-1]$

VI. Rayleigh Power Method:

Find AX_i and express $AX_i = |\lambda_i| X_{i+1}$ where $|\lambda_i|$ is the numerically largest element of X_i. Iterate until desired accuracy.

The largest/dominant eigenvalue is given by $\lambda = \dfrac{\langle AX_n, X_n \rangle}{\langle X_n, X_n \rangle} = \dfrac{X_n^T (AX_n)}{X_n^T X_n}$.

Find the dominant eigenvalue and the corresponding eigenvector of the following matrices.

1. $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ by taking the initial vector as $[1 \quad 0 \quad 0]^T$.

2. $\begin{bmatrix} -0.5 & 0.5 & -1 \\ 0.5 & -0.5 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ by taking the initial vector as $[1 \quad 1 \quad 1]^T$. Ans: $\lambda \approx\sim -2$. Others are 1 and 0.

3. $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ by taking the initial vector as $[1 \quad 0 \quad 0]^T$.

4. $\begin{bmatrix} -8 & 6 & -2 \\ 6 & -7 & 4 \\ -2 & 4 & -3 \end{bmatrix}$ by taking the initial vector as $[1 \quad 1 \quad 1]^T$. Ans: $\lambda \approx \sim -15$.

5. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ by taking the initial vector as $[1 \quad 0 \quad 0]^T$.

6. $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ by taking the initial vector as a) $[1 \quad 0 \quad 0]^T$ and b) $[1 \quad 1 \quad 1]^T$.

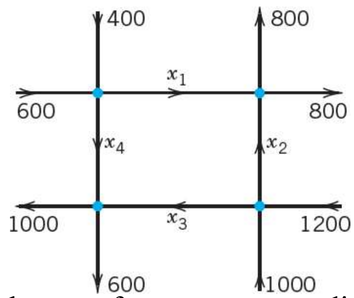
7. $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ by taking the initial vector as $[1 \quad 1 \quad 1]^T$.

8. $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ by taking the initial vector as $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$.

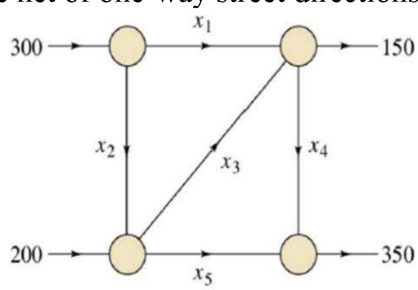
VII. APPLICATIONS:

TRAFFIC FLOW PROBLEMS

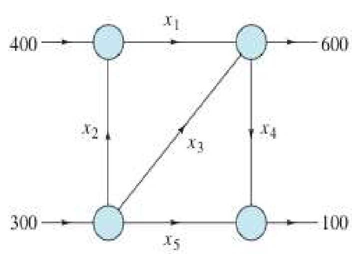
1. Find the traffic flow in the net of one-way street directions shown in the figure:



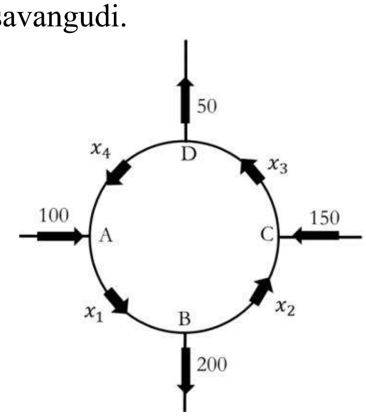
2. Find the traffic flow in the net of one-way street directions shown in the figure:



3. The flow of traffic (in vehicles per hour) through a network of streets is shown in the figure.
   i. Solve this system for x_i, i=1, 2, ……..5
   ii. Find the traffic flow when $x_3=0$ and $x_5=0$.
   iii. Find the traffic flow when $x_3=x_5=100$



4. Following figure represents traffic entering and leaving a round of road junction, such junctions are common in Basavangudi.



(i) Construct a mathematical model that describe the flow of traffic along various branches.
(ii) What is the minimum flow theoretically possible along branch AB.? What are the other flows at that time?
(iii) What is the minimum flow theoretically possible along branch DA if the branch BC is under repair? What is the other flow at that time?
(iv) If it possible to repair branch CD without having another alternative road.

BALANCING CHEMICAL EQUATIONS

Balance the following chemical equations using matrix method.

1. $B_2S_3 + H_2O \rightarrow H_3BO_3 + H_2S$
2. $C_2H_6 + O_2 \rightarrow CO_2 + H_2O$
3. $KClO_3 \rightarrow KCl + O_2$
4. $NA + H_2O \rightarrow NaOH + H_2$