



B.M.S. COLLEGE OF ENGINEERING, BENGALURU – 560 019

Autonomous college, affiliated to VTU

DEPARTMENT OF MATHEMATICS

Unit 4: Ordinary Differential Equations of First Order

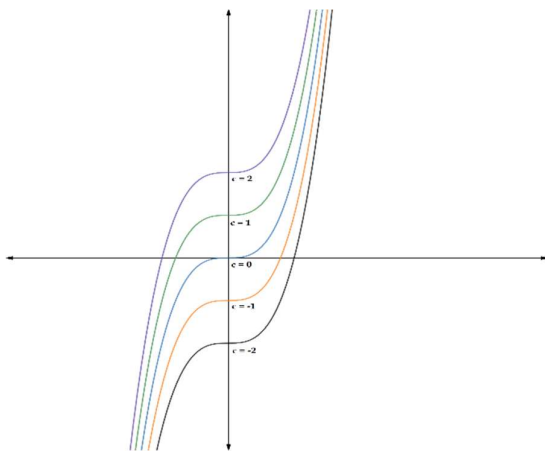
Courses: *Mathematical Foundation for Computer Science Stream-I (25MA1BSMCS)*

Mathematical Foundation for Civil, Electrical and Mechanical Streams– I (25MA1BSCM)

Differential equations

Any equation which involves derivatives, dependent variables and independent variables is called a differential equation.

Classification	
Ordinary differential equations	Linear differential equations
	Nonlinear differential equations
Partial differential equations	Linear, semi linear, quasi linear ...



Solution of a differential equation: A relation between x and y which satisfies the given differential equation.

General Solution: A solution which contains arbitrary constants.

Solution of first order ODE: Geometrically, the solution of a first order ODE represents a family of curves.

Linear Ordinary differential equations of 1st order

An equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$ is called a linear first order differential equation.

The solution to this equation is given by $y(IF) = \int (IF)Qdx + c$ where $IF = e^{\int Pdx}$ and c is the constant of integration.

I. BERNOULLI'S EQUATION

The equation of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ is called the Bernoulli's equation. It can be reduced to a linear differential equation by dividing throughout by y^n , to get $y^{-n} \frac{dy}{dx} + Py^{1-n} = Q$ and substituting $y^{1-n} = z$ the equation becomes $\frac{dz}{dx} + P(1-n)z = Q(1-n)$ which is linear in z .



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Solve the following differential equations:

1. $x \frac{dy}{dx} + y = x^3 y^6$.
2. $y' + \frac{y}{2x} = \frac{x}{y^3}, y(1) = 2$.
3. $(y \log x - 2) y dx - x dy = 0$.
4. $(x^3 y^2 + xy) dx = dy$.
5. $\cos x dy = y(\sin x - y) dx$.
6. $y^2 y' - y^3 \tan x - \sin x \cos^2 x = 0$.
7. $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$.
8. $3y' + xy = xy^{-2}$.
9. $\frac{dy}{dx} = \frac{y}{x + \sqrt{xy}}$.
10. $\frac{dz}{dx} + \frac{z}{x} \log z = \frac{z}{x} (\log z)^2$.
11. Suppose a student carrying a flu virus returns to an isolated college of 1000 students. It is assumed that the rate at which the virus spreads is proportional not only to the number x of the infected students but also to the number of students not infected. The equation is given by $\frac{dx}{dt} = kx(1000 - x), x(0) = 1$. Determine the number of infected students after 6 days if it is further observed that after 4 days, $x(4) = 50$ by treating the equation as Bernoulli's equation.

II. EXACT DIFFERENTIAL EQUATIONS

A differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is said to be exact if $M(x, y)dx + N(x, y)dy$ is the exact differential of some function $u(x, y)$ i.e. $du = Mdx + Ndy$. Its solution, therefore is $u(x, y) = c$.



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The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, and its solution is given by

$$\int_{(y \text{ constant})} Mdx + \int (\text{terms of } N \text{ without } x) dy = c.$$

Solve the following differential equations:

1. $(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0.$
2. $\left[y \left(1 + \frac{1}{x} \right) + \cos y \right] dx + (x + \log x - x \sin y) dy = 0.$
3. $(2xy + y - \tan y)dx + (x^2 - x \tan^2 y + \sec^2 y)dy = 0.$
4. $\frac{2x}{y^3} dx + \frac{y^2 - 3x^2}{y^4} dy = 0.$
5. $y' = \frac{y - 2x}{2y - x}, y(1) = 2.$
6. $(y^2 e^{xy^2} + 4x^3)dx + (2xy e^{xy^2} - 3y^2)dy = 0.$
7. $(\sec x \tan x \tan y - e^x)dx + \sec x \sec^2 y dy = 0.$
8. $(y \sin 2x)dx - (1 + y^2 + \cos^2 x)dy = 0.$
9. $(\cos x - x \cos y)dy - (\sin y + y \sin x)dx = 0.$
10. $\left(\frac{y}{(x+y)^2} - 1 \right) dx + \left(1 - \frac{x}{(x+y)^2} \right) dy = 0.$
11. $\sin x \cosh y dx - \cos x \sinh y dy = 0, y(0) = 3.$

III. DIFFERENTIAL EQUATIONS REDUCIBLE TO THE EXACT FORM

Type 1: In the differential equation $Mdx + Ndy = 0$, if $\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)}{N}$ a function of x say $f(x)$,

then $IF = e^{\int f(x) dx}$.



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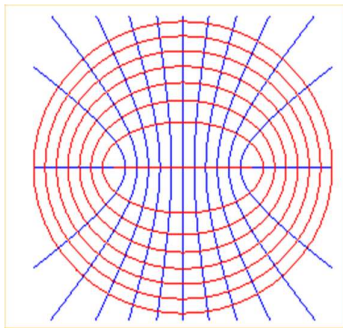
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Type 2: In the differential equation $Mdx + Ndy = 0$, if $\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{-M}$ a function of y say $F(y)$,
then $IF = e^{\int F(y)dy}$.

Solve the following differential equations by reducing it to exact differential equations:

1. $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$
2. $y \log y dx + (x - \log y)dy = 0$
3. $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2ydy = 0$
4. $y(2x^2 - xy + 1)dx + (x - y)dy = 0$
5. $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$.
6. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$.
7. $y(x + y)dx + (x + 2y - 1)dy = 0$.
8. $2xydy - (x^2 + y^2 + 1)dx = 0$.
9. $3(x^2 + y^2)dx + x(x^2 + 3y^2 + 6y)dy = 0$.
10. $2xydx + (y^2 - x^2)dy = 0$.

IV. ORTHOGONAL TRAJECTORIES



Consider two families of plane curves F_1 and F_2 . If each member of F_1 intersects each member of F_2 at right angles, then the family of curves F_2 are said to be orthogonal trajectories of the family of curves F_1 and vice-versa.

Orthogonal trajectories in Cartesian form: Working Rule

- ❖ Obtain the differential equation of the given family of curves.
- ❖ Replace $\frac{dy}{dx}$ by $\left(-\frac{dx}{dy}\right)$ in the differential equation obtained in the previous step.
- ❖ This generates the differential equation for the orthogonal trajectories.
- ❖ The general solution to the differential equation of the orthogonal trajectories gives the required family of orthogonal trajectories.



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1. Find the orthogonal trajectories of the family of:
a) $y^2 = 4ax$. b) $y = c(\sec x + \tan x)$. c) $y = ax^2$.
d) $x^2 - y^2 = c$. e) $x^2 - y^2 = cx$.
2. The electric lines of force of two opposite charges of the same strength at $(1,0)$ and $(-1,0)$ are circles (through these points) of the form $x^2 + y^2 - ay = 1$. Find their equipotential lines (orthogonal trajectories)
3. Find the orthogonal trajectories of the family of confocal conics $\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1$, where λ is the parameter.
4. Given $y = ce^{-2x} + 3x$, find that member of the orthogonal trajectory which passes through point $(0, 3)$.
5. The electric field between two concentric cylinders, the equipotential lines (lines of constant potential) are circles given by $U(x, y) = x^2 + y^2 = \text{constant}(\text{volts})$, what are their orthogonal trajectories (the curves of electric force)?
6. If the streamlines of the flow are (paths of the particles of the fluid) in the channel are $\psi(x, y) = xy = \text{constant}$, what their orthogonal trajectories (called equipotential lines) are?
7. If the isotherms (curves of constant temperature) in a body are $T(x, y) = 2x^2 + y^2 = C$, a constant, what are their orthogonal trajectories (curves along which heat will be flowing in regions free of heat sources or sinks and filled with homogeneous material)?
8. Isotherms of a lamina in the xy -plane are defined by $x^2 + 3y^2 = c$. What are their orthogonal trajectories (flux lines). Then determine the isotherm and the flux line passing through
a) $(0,2)$ b) $(-1,1)$ c) $(2,0)$.
9. Show that the family of curves are self-orthogonal:
a) $x^2 = 4a(y + a)$.
b) Confocal conics, $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1$, λ being the parameter.
c) $y^2 = 4a(x + a)$.

V. **APPLICATIONS: Newton's law of Cooling**

The temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.



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If q_0 is the temperature of the surroundings and q is the temperature of the body at any time t , then

$$\frac{dq}{dt} = -k(q - q_0), \text{ where } k \text{ is a constant.}$$

1. When a cake is removed from an oven, its temperature is measured at 300F. Three minutes later its temperature is 200°F. How long will it take for the cake to cool off to a temperature of 80 degrees? The room temperature is 70°F.

Ans: 16.49 minutes

2. A thermometer is removed from a room where the temperature is 70°F and is taken outside, where the air temperature is 10°F. After one-half minute the thermometer reads 50°F. What is the reading of the thermometer at t 1 min? How long will it take for the thermometer to reach 15°F?

Ans: T(1) = 36.67°F and t = 3.06 minutes

3. If the temperature of the air is 30°C and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will 40°C.

Ans: 52 minutes

4. A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C. What will be the temperature of the body after 40 minutes from the original?

Ans: 40 minutes

5. A dead body was found within a closed room of a house where the temperature was a constant 70°F. At the time of discovery, the core temperature of the body was determined to be 85°F. One hour later a second measurement showed that the core temperature of the body was 80°F. Assume that the time of death corresponds to $t = 0$ and that the core temperature at that time was 98.6°F. Determine how many hours elapsed before the body was found. **Ans: 1.5916 hours = 1 hour 35 min 30 sec**
6. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 24 minutes.

Ans: T(24) = 48°C