For the Course Code: 23MA1BSCEM, 23MA1BSMCS

Differential equations:
Any equation which involves derivatives, dependent variables and independent variables is called a differential equation.

Classification	
Ordinary differential equations	Linear differential equations Nonlinear differential equations
Partial differential equations	Linear, semi linear, quasi linear …

Solution of a differential equation: A relation between x and y which satisfies the given differential equation.

General Solution: A solution which contains arbitrary constants.

Solution of first order ODE: Geometrically, the solution of a first order ODE represents a family of curves.

Linear Ordinary differential equations of 1st order

An equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$ is called a linear first order differential equation.

The solution is given by $y(IF)=\int(IF)Q+c$ where $IF=e^{\int Pdx}$ and c is the constant of integration.

I. BERNOULLI'S EQUATION

The equation of the form $\frac{dy}{dx}+P(x)y=Q(x)y^n$ is called the Bernoulli's equation. It can be reduced to a linear differential equation by dividing throughout by y^n, to get $y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$ and substituting $y^{1-n}=z$.

<u>EXAMPLES:</u>

Solve the following differential equations:

1 $x\frac{dy}{dx}+y=x^3y^6$.

2 $y'+\frac{y}{2x}=\frac{x}{y^3}, y(1)=2$.

3 $(y\log x-2)ydx-xdy=0$.

4 $\left(x^3y^2+xy\right)dx=dy.$

5 $\cos x dy = y(\sin x - y)dx.$

6 $r\sin\theta-\cos\theta\dfrac{dr}{d\theta}=r^2.$

7 $y^2y'-y^3\tan x-\sin x\cos^2 x=0.$

8 $\dfrac{dy}{dx}=\dfrac{x^2+y^2+1}{2xy}.$

9 $3y'+xy=xy^{-2}.$

10 $\dfrac{dy}{dx}=\dfrac{y}{x+\sqrt{xy}}.$

11 $\dfrac{dz}{dx}+\dfrac{z}{x}\log z=\dfrac{z}{x}\left(\log z\right)^2.$

II. EXACT DIFFERENTIAL EQUATIONS

A differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is said to be exact if $M(x,y)dx+N(x,y)dy$ is the exact differential of some function $u(x,y)$ i.e. $du=Mdx+Ndy.$ Its solution, therefore is $u(x,y)=c$.

The necessary and sufficient condition for the differential equation $Mdx+Ndy=0$ to be exact is $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$, and its solution is given by $\displaystyle\int_{(y\text{ constant})} Mdx+\int\left(\text{terms of } N \text{ without } x\right)dy=c$.

<u>EXAMPLES:</u>

Solve the following differential equations:

1 $\left(x^4-2xy^2+y^4\right)dx-\left(2x^2y-4xy^3+\sin y\right)dy=0.$

2 $\left[y\left(1+\dfrac{1}{x}\right)+\cos y\right]dx+\left(x+\log x-x\sin y\right)dy=0.$

3 $\left(2xy+y-\tan y\right)dx+\left(x^2-x\tan^2 y+\sec^2 y\right)dy=0.$

4 $\dfrac{2x}{y^3}dx+\dfrac{y^2-3x^2}{y^4}dy=0.$

5 $y'=\dfrac{y-2x}{2y-x},\ \mathrm{y}(1)=2.$

6 $\left(y^2e^{xy^2}+4x^3\right)dx+\left(2xye^{xy^2}-3y^2\right)dy=0$.

7 $\left(\sec x\tan x\tan y-e^x\right)dx+\sec x\sec^2 y\,dy=0$.

8 $\left(y\sin 2x\right)dx-\left(1+y^2+\cos^2 x\right)dy=0$.

9 $\left(\cos x-x\cos y\right)dy-\left(\sin y+y\sin x\right)dx=0$.

10 $\left(\dfrac{y}{(x+y)^2}-1\right)dx+\left(1-\dfrac{x}{(x+y)^2}\right)dy=0$.

11 $\sin x\cosh y\,dx-\cos x\sinh y\,dy=0,\ y(0)=3$.

III. DIFFERENTIAL EQUATIONS REDUCIBLE TO THE EXACT FORM

Type 1: In the differential equation $Mdx+Ndy=0$, if $\dfrac{\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)}{N}$ a function of x say $f(x)$, then $IF=e^{\int f(x)dx}$.

<u>EXAMPLES:</u>

Solve the following differential equations by reducing it to exact differential equations

1 $\left(xy^2-e^{1/x^3}\right)dx-x^2y\,dy=0$.

2 $y(x+y)dx+(x+2y-1)dy=0$.

3 $\left(2y^2+3x\right)dx+2xy\,dy=0$.

4 $\left(x^2+y^2-5\right)dx=\left(y+xy\right)dy,\quad y(0)=1$.

Type 2: In the differential equation $Mdx+Ndy=0$, if $\dfrac{\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)}{-M}$ a function of y say $F(y)$, then $IF=e^{\int F(y)dy}$.

<u>EXAMPLES:</u>

Solve the following differential equations by reducing it to exact differential equations

1 $\left(3x^2y^4+2xy\right)dx+\left(2x^3y^3-x^2\right)dy=0$.

2 $\left(xy^3+y\right)dx+2\left(x^2y^2+x+y^4\right)dy=0.$

3 $3\left(x^2+y^2\right)dx+x\left(x^2+3y^2+6y\right)dy=0.$

4 $6xydx+(4y+9x^2)dy=0.$

5 $xdx+\left(x^2y+4y\right)dy=0,\ y(4)=0.$

APPLICATIONS

MIXING PROBLEM

EXAMPLES:

1 A tank is initially filled with 100 gallons of salt solution containing 1 lb of salt per gallon. Fresh brine containing 2 lb per gallon of salt runs into the tank at the rate of 5 gallons per minute and the mixture, assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt at any time, and determine how long will it take for this amount to reach 150 lb.

2 Suppose a large mixing tank initially holds 300 gallons of brine. Another brine solution is pumped into the tank at a rate of 3gallons per minute; the concentration of the salt in this inflow is 2 lb/gal of salt. When the solution is stirred well, it is pumped out at the same rate. If there were 50 lb of salt dissolved initially in the 300 gallons, how much salt is there in the long time?

3 A tank contains 200 litres of fluid in which 30 grams of salt is dissolved. Brine containing 1 gram of salt per litre is then pumped into the tank at a rate of 4 L/min.; the well mixed solution is pumped out at the same rate. Find the amount of salt in the tank at time t.

4 A tank contains 1000 gallons of brine in which 500 lb of salt are dissolved. Fresh water runs into the tank at the rate of 10 gallons per minute and the mixture is kept uniform by stirring, runs out at the same rate. How long will it be before only 50 lb of salt is left in the tank?

ORTHOGONAL TRAJECTORIES

In Cartesian Co-ordinates:

EXAMPLES:

Find the orthogonal trajectories of the family of:

a) $y=c\left(\sec x+\tan x\right).$

b) $y^2=4ax.$

c) $x^2-y^2=cx$.

d) $ay^2=x^3.$

e) Find the orthogonal trajectory of the family of confocal conics

$\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1, \lambda$ being the parameter.

f) Show that the family of curves are self-orthogonal:

a) Confocal conics, $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \lambda$ being the parameter.

b) $y^2=4a(x+a)$.

c) $y^2=2cx+c^2$.

In Polar coordinates:

EXAMPLES:

Find the orthogonal trajectories (O.T.) of each of the following family of curves:

a) $r=\frac{2a}{(1+\cos\theta)}$

b) $r^2=a^2\cos 2\theta$

c) $r=4a\sec\theta\cdot\tan\theta$

d) $r=a\left(1+\sin^2\theta\right)$

e) $r=a\sin\theta\tan\theta$

f) $r=a(\sec\theta+\tan\theta)$

g) $r^n\cos n\theta=a^n$