



BMS COLLEGE OF ENGINEERING, BENGALURU-19
Autonomous Institute, Affiliated to VTU

DEPARTMENT OF MATHEMATICS

Dept. of Maths., BMSCE

Unit - 2: Multivariable Calculus

For the Course Code: 23MA1BSCem, 23MA1BSMCS

I. PARTIAL DERIVATIVES

Let $u = f(x, y)$ be a function of two variables x and y . If we keep y as constant and vary x alone, then u is a function of x only. The derivative of u with respect to x , treating y as constant is called the partial derivative of u w. r. t x and is denoted by one of the symbols $\frac{\partial u}{\partial x}$, u_x . Thus $\frac{\partial u}{\partial x} = u_x = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$. Similarly $\frac{\partial u}{\partial y} = u_y = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$. Further u_x and u_y are also functions of x and y , so these can further be differentiated partially w. r. t. x and y . Thus $\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial x^2}$ or u_{xx} , $\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial x \partial y}$ or u_{xy} , $\frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial y \partial x}$ or u_{yx} and $\frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial^2 u}{\partial y^2}$ or u_{yy} .

Examples

1. Prove that $y v_y - x v_x = -y^2 v^3$ if $v = (1 - 2xy + y^2)^{-1/2}$.
2. Show that $w_x + w_y + w_z = 0$ if $w = (y - z)(z - x)(x - y)$.
3. Show that $u_x + u_y = u$, if $u = \frac{e^{x+y}}{e^x + e^y}$.
4. If $w = x^2 y + y^2 z + z^2 x$, prove that $w_x + w_y + w_z = (x + y + z)^2$.
5. If $u = \frac{y}{z} + \frac{z}{x}$, show that $x u_x + y u_y + z u_z = 0$.
6. Given $u = e^{r \cos \theta} \cos(r \sin \theta)$, $v = e^{r \cos \theta} \sin(r \sin \theta)$, Prove that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = \frac{-1}{r} \frac{\partial u}{\partial \theta}.$$

7. If $(x + y)z = x^2 + y^2$, Show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)^2 = 4 \left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right]$.
8. Show that $(v_x)^2 + (v_y)^2 + (v_z)^2 = v^4$ if $v = (x^2 + y^2 + z^2)^{-1/2}$.
9. If $z = e^{ax+by} f(ax - by)$ prove that $b \frac{\partial z}{\partial x} + a \frac{\partial z}{\partial y} = 2abz$.



10. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, then show that

$$(i) \frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2} \text{ and } (ii) \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}.$$

11. If $u = \log_e(x^3 + y^3 - x^2y - xy^2)$, then show that $u_{xx} + 2u_{xy} + u_{yy} = -\frac{4}{(x+y)^2}$.

12. If $z = f(x+ct) + \phi(x-ct)$, then prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$.

13. If $u = e^{a\theta} \cos(a \log_e r)$ Show that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$.

14. Prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, if

a) $u = x^y$

b) $u = \log_e\left(\frac{x^2 + y^2}{xy}\right)$

15. Verify that $f_{xy} = f_{yx}$ when $f(x, y) = \sin^{-1}(y/x)$.

16. Prove that $u_{xx} + u_{yy} = 0$, if

a) $u = \tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)$

b) $u = \log_e(x^2 + y^2) + \tan^{-1}(y/x)$

17. Find the value of n so that the equation $v = r^n(3\cos^2\theta - 1)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta}\right) = 0.$$

18. If $Z = \tan(y+ax) + (y-ax)^{3/2}$, Show that $Z_{xx} = a^2 Z_{yy}$.

19. If $v = \frac{1}{\sqrt{t}} e^{-x^2/4a^2t}$, Prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$.

20. If $\theta = t^n e^{-r^2/4t}$, what value of n will make $\frac{1}{r^2} \left[\frac{\partial}{\partial r} \left(r^2 \frac{\partial \theta}{\partial r} \right) \right] = \frac{\partial \theta}{\partial t}$.

21. If $x^x y^y z^z = c$, show that $z_{xy} = -[x \log_e(e^x)]^{-1}$, when $x = y = z$.

22. If $z = \log_e(e^x + e^y)$, Show that $rt - s^2 = 0$ where $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$.

23. If $v = \log_e(x^2 + y^2 + z^2)$, prove that $(x^2 + y^2 + z^2)[v_{xx} + v_{yy} + v_{zz}] = 2$.

24. If $u = \log_e[x^3 + y^3 + z^3 - 3xyz]$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

25. If $w = r^m$, prove that $w_{xx} + w_{yy} + w_{zz} = m(m+1)r^{m-2}$ where $r^2 = x^2 + y^2 + z^2$.



26. Verify that v satisfies Laplace's equation $v_{xx} + v_{yy} + v_{zz} = 0$ if

- a) $v = e^{3x+4y} \cos 5z$
- b) $v = (x^2 + y^2 + z^2)^{-1/2}$
- c) $v = \cos 3x \cos 4y \sinh 5z$

II. COMPOSITE FUNCTIONS

Total Derivative:

If $u = f(x, y)$ where $x = \phi(t)$ and $y = \psi(t)$ then $\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$.

Examples

1. Find the differential of the following functions:

a) $f(x, y) = x \cos y - y \cos x$

2. $f(x, y) = e^{xyz}$

3. Find $\frac{du}{dt}$ for the following functions:

a) $u = x^2 - y^2$, $x = e^t \cos t$, $y = e^t \sin t$ at $t = 0$.

b) $u = x^y$, when $y = \tan^{-1} t$, $x = \sin t$

c) $u = \sin(e^x + y)$, $x = f(t)$, $y = g(t)$

d) $u = \tan^{-1}(y/x)$, $x = e^t - e^{-t}$, $y = e^t + e^{-t}$

e) $u = xy + yz + zx$, $x = 1/t$, $y = e^t$ and $z = e^{-t}$

f) $u = x^3 y e^z$ where $x = t$, $y = t^2$ and $z = \log_e t$, at $t = 2$.

4. Find $\frac{du}{dt}$ and verify the result by direct substitution.

a) $u = \sin\left(\frac{x}{y}\right)$ and $x = e^t$, $y = t^2$.

b) $u = x^2 + y^2 + z^2$, $x = e^{2t}$, $y = e^{2t} \cos 3t$ and $z = e^{2t} \sin 3t$

5. If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, show that $\frac{du}{dt} = 3(1 - t^2)^{-\frac{1}{2}}$. Also verify the result by direct substitution.

6. The altitude of a right circular cone is 15cm and is increasing at 0.2cm/s. The radius of the base is 10cm and is decreasing at 0.3cm/s. How fast is the volume changing?



- Find the rate at which the area of a rectangle is increasing at a given instant when the sides of the rectangle are 4 ft and 3 ft and are increasing at the rate of 1.5 fts^{-1} and 0.5 fts^{-1} respectively.
- In order that the function $u = 2xy - 3x^2y$ remains constant, what should be the rate of change of y w.r.t. t , given x increases at the rate of 2cm/sec at the instant when $x = 3\text{cm}$ and $y = 1\text{cm}$.

Partial differentiation of Composite functions:

If $u = f(x, y)$ where $x = g(s, t)$ and $y = h(s, t)$ then

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$$

Examples

- If $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$ find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$.
- If $W = u^2v$ and $u = e^{x^2 - y^2}$, $v = \sin(xy^2)$ find $\frac{\partial W}{\partial x}$ and $\frac{\partial W}{\partial y}$.
- If $u = u\left[\frac{y-x}{xy}, \frac{z-x}{xz}\right]$, show that $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$.
- If $u = F(x - y, y - z, z - x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.
- Prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$ if z is a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$.
- If $V = f(r, s, t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$ show that $x \frac{\partial V}{\partial x} + y \frac{\partial V}{\partial y} + z \frac{\partial V}{\partial z} = 0$.
- If $x = u + v + w$, $y = vw + wu + uv$, $z = uvw$ and F is a function of x, y, z show that $u \frac{\partial F}{\partial u} + v \frac{\partial F}{\partial v} + w \frac{\partial F}{\partial w} = x \frac{\partial F}{\partial x} + 2y \frac{\partial F}{\partial y} + 3z \frac{\partial F}{\partial z}$.
- If z is a function of x and y and $x = e^u \cos v$, $y = e^u \sin v$.

Prove that (i) $x \frac{\partial z}{\partial v} + y \frac{\partial z}{\partial u} = e^{2u} \frac{\partial z}{\partial y}$ (ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u} \left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

**III. JACOBIAN**

If u and v are functions of two independent variable x and y then the Jacobian of u, v w.r.to x, y is denoted by

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ or } J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Properties of Jacobians

1. If u, v are functions of x, y and x, y are functions of u, v

$$J = \frac{\partial(u,v)}{\partial(x,y)} \text{ and } J' = \frac{\partial(x,y)}{\partial(u,v)} \text{ then } JJ' = 1.$$

2. If u, v are functions of r, s and r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}.$$

Examples

- Find $\frac{\partial(u,v)}{\partial(x,y)}$ for the following:
 - $u = e^x \sin y, v = x + \log_e(\sin y).$
 - $u = x + y, y = uv$
- If $x = a \cosh \xi \cos \eta, y = a \sinh \xi \sin \eta$ Show that $\frac{\partial(x,y)}{\partial(\xi,\eta)} = \frac{1}{2}a^2(\cosh 2\xi - \cos 2\eta).$
- Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ for the following:
 - $u = x^2, v = \sin y, w = e^{-3z}.$
 - $u = x^2 + y^2 + z^2, v = xy + yz + zx, w = x + y + z.$
 - $u = x + 3y^2 - z^3, v = 4x^2yz, w = 2z^2 - xy$ at $(1, -1, 0).$
 - $u = \frac{2yz}{x}, v = \frac{3zx}{y}, w = \frac{4xy}{z}$
- If $y_1 = \frac{x_2 x_3}{x_1}, y_2 = \frac{x_3 x_1}{x_2}, y_3 = \frac{x_1 x_2}{x_3}$, then show that $\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4.$
- Find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if
 - $x = \frac{u^2 - v^2}{2}, y = uv, z = w.$
 - $u = x + y + z, uv = y + z, uvw = z,$
- Verify that $JJ' = 1$
 - $x = e^u \cos v, y = e^u \sin v.$



- b) $x = u(1-v), y = uv.$
- c) $u = x + \frac{y^2}{x}, v = \frac{y^2}{x}.$
- d) $x = u, y = u \tan v, z = w$
7. Find $\frac{\partial(u,v)}{\partial(r,\theta)}$ when,
- a) $u = x^2 - 2y^2, v = 2x^2 - y^2$ and $x = r \cos \theta, y = r \sin \theta$
- b) $u = x^2 - y^2, v = 2xy$ and $x = r \cos \theta, y = r \sin \theta$
- c) $u = 2xy, v = x^2 - y^2$ and $x = r \cos \theta, y = r \sin \theta$
- d) $u = 2axy$ & $v = a(x^2 - y^2)$ and $x = r \cos \theta, y = r \sin \theta$
8. Find $\frac{\partial(X,Y)}{\partial(x,y)}$ where $X = u^2v, Y = uv^2$ and $u = x^2 - y^2, v = yx$
9. If $x = \sqrt{vw}, y = \sqrt{wu}, z = \sqrt{uv}$ and $u = r \sin \theta \cos \phi, v = r \sin \theta \sin \phi, w = r \cos \theta,$ calculate $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}.$
10. If $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}, v = \sin^{-1} x + \sin^{-1} y,$ show that u & v are functionally dependent and find the functional relationship.
11. If $u = \frac{x+y}{1-xy}, v = \tan^{-1} x + \tan^{-1} y,$ show that u & v are functionally dependent and find the functional relationship.

III. TAYLOR'S SERIES FOR FUNCTION OF TWO VARIABLES

$$f(x, y) = f(a, b) + \Delta f(a, b) + \frac{\Delta^2 f(a, b)}{2!} + \frac{\Delta^3 f(a, b)}{3!} + \dots \text{ where}$$

$$\Delta = (x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y}.$$

Examples

Expand the following functions upto second degree terms.

- $f(x, y) = x^2 + xy + y^2$ in powers of $(x-1)$ and $(y-2).$
- $f(x, y) = (1+x+y)^{-1}$ in powers of $(x-1)$ and $(y-1).$
- $f(x, y) = e^x \cos y$ in powers of $(x-1)$ and $(y-\pi/4).$
- $f(x, y) = x^y$ in powers of $(x-1)$ and $(y-1)$ and also find $(1.1)^{1.1}.$
- $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ in powers of $(x-1)$ and $(y-1).$ Hence compute $f(1.1, 0.9).$



6. $f(x, y) = x^3 + xy^2 + y^3$ in powers of $(x-1)$ and $(y-2)$.
7. $f(x, y) = \cot^{-1}(xy)$ in powers of $(x+0.5)$ and $(y-2)$ and hence compute $f(-0.4, 2.2)$.
8. $f(x, y) = x^2y + 3y - 2$ in powers of $(x+1)$ and $(y-2)$.
9. $f(x, y) = \sin(xy)$ in powers of $(x-1)$ and $(y-\pi/2)$
10. $f(x, y) = xy^2 + \cos xy$ in powers of $(x-1)$ and $(y-\pi/2)$.
11. $f(x, y) = x^2y + \sin y + e^x$ about $(1, \pi)$.
12. $f(x, y) = e^x \sin y$ about $(-1, \pi/4)$.

V. MACLAURIN'S SERIES FOR FUNCTION OF TWO VARIABLES

$$f(x, y) = f(0, 0) + \Delta f(0, 0) + \frac{\Delta^2 f(0, 0)}{2!} + \frac{\Delta^3 f(0, 0)}{3!} + \dots \text{ where } \Delta = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}.$$

Examples

1. $\cos x \cos y$ in powers of x and y up to second degree terms.
 2. Expand $e^y \log_e(1+x)$ in powers of x and y up to third degree terms.
 3. Expand $e^{ax} \sin by$ in the ascending powers of x and y up to third degree terms.
 4. Find the Maclaurin's expansion of $e^x \log_e(1+y)$ up to third degree terms.
- Expand $\log_e(1+x+y)$ in the neighbourhood of $(0,0)$ up to second degree terms.

IV. MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

A function $z = f(x, y)$ is said to have a maximum or minimum at $x=a, y=b$ as

$$f(a+h, b+k) < f(a, b) \text{ or } f(a+h, b+k) > f(a, b) \text{ for all values of } h \text{ and } k.$$

Necessary conditions for $f(x, y)$ to have a maximum or a minimum at (a, b) are

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0.$$

Working rule to find the maxima and minima of $z = f(x, y)$

- 1) Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ and equate them to zero, solve these as simultaneous equations in x and y . Let $(a, b), (c, d), \dots$ be the roots of the simultaneous equations.
- 2) Calculate the values of $r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$ for each of points.
- 3) If $rt - s^2 > 0$ and $r < 0$ at (a, b) , $f(x, y)$ has a maximum value.



- 4) If $rt - s^2 > 0$ and $r > 0$ at (a, b) , $f(x, y)$ has a minimum value.
- 5) If $rt - s^2 < 0$ at (a, b) , (a, b) is a saddle point.
- 6) If $rt - s^2 = 0$ at (a, b) then the case is doubtful and needs further investigation.

Examples

1. Find the extreme values of the following functions:
 - a) $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.
 - b) $f(x, y) = 2(x^2 - y^2) - x^4 + y^4$.
 - c) $f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$
 - d) $x^4 + y^4 - x^2 - y^2 + 1$
 - e) $x^2 + 2y^2 + 3z^2 - 2xy - 2yz - 2$
 - f) $f(x, y) = x^3 y^2 (1 - x - y)$
 - g) $f(x, y) = x^3 + y^3 - 3y - 12x + 20$
 - h) $f(x, y) = \cos x \cos y \cos(x + y)$
 - i) $x^2 + y^2 + 6x + 12 = 0$
 - j) $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$
2. Find the shortest distance from origin to the plane $x - 2y - 2z = 3$.
3. Find the points on the surface $z^2 = xy + 1$ nearest to the origin.
4. Find the shortest distance from origin to the surface $xyz^2 = 2$.
5. Sum of three numbers is a constant. Prove that their product is maximum when they are equal.
6. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.
7. Examine the function $f(x, y) = \sin x + \sin y + \sin(x + y)$, $x, y \in (0, \pi)$ for extreme values.
8. In a plane triangle find the maximum value of $\cos A \cos B \cos C$ where A, B and C are the angles of the triangle.
9. Find the minimum value of $x^2 + y^2 + z^2$ given $ax + by + cz = p$
10. A flat circular plate is heated so that the temperature at any point (x, y) is $u(x, y) = x^2 + 2y^2 - x$. Find the coldest point on the plate.
11. A rectangular box open at the top is to have a volume 108 cubic meters. Find its dimensions if its total surface area is minimum.



12. The temperature T at any point (x, y, z) in space is $T(x, y, z) = kxyz^2$ where k is a constant. Find the highest temperature on the surface of the sphere $x^2 + y^2 + z^2 = a^2$.