



**B.M.S. COLLEGE OF ENGINEERING
DEPARTMENT OF MATHEMATICS**

Dept. of Math., BMSCE

Unit 2: Calculus of One Variable

Course Code: 25MA1BSCEM

Course: Mathematical Foundation for Civil, Electrical and Mechanical Streams– 1

UNIT-2: Calculus of One Variable

Polar Curves:

Angle between radius vector and tangent:

If $r = f(\theta)$, then the angle between radius vector and tangent is given by

$$\tan \phi = r \frac{d\theta}{dr}.$$

1. If ϕ be the angle between radius vector and the tangent at any point of the curve $r = f(\theta)$

then prove that $\tan(\phi) = r \frac{d\theta}{dr}$.

2. Find the angle between the radius vector and the tangent for the following polar curves.

a) $r = a(1 + \cos \theta)$

Ans: $\pi/2 + \theta/2$.

b) $r^2 = a^2 \sin^2 \theta$

Ans: $\phi = \theta$

c) $\frac{l}{r} = 1 + e \cos \theta$

Ans: $\phi = \tan^{-1} \left[\frac{1 + e \cos \theta}{e \sin \theta} \right]$.

d) $r^m \cos m\theta = a^m$

Ans: $\pi/2 - m\theta$

3. Find the angle between the radius vector and the tangent for the following polar curves. And also find slope of the tangent at the given point.

e) $\frac{2a}{r} = 1 - \cos \theta$ at $\theta = 2\pi/3$

Ans: $\phi = \frac{2\pi}{3}, \tan \psi = \sqrt{3},$.

f) $r \cos^2(\theta/2) = a^2$ at $\theta = 2\pi/3$

Ans: $\phi = \frac{\pi}{6}$.

g) $r^2 \cos(2\theta) = a$

Ans: $\phi = \frac{\pi}{2} - 2\theta; \psi = \frac{\pi}{2} - \theta$

Angle between curves:

Angle of intersection of two polar curves = angle of intersection of their tangents denoted by α

$$\alpha = |\phi_2 - \phi_1| \quad \text{or} \quad \tan \alpha = \left| \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \tan \phi_2} \right|$$



**B.M.S. COLLEGE OF ENGINEERING
DEPARTMENT OF MATHEMATICS**

Dept. of Math., BMSCE

Unit 2: Calculus of One Variable

4. Find the angle of intersection of the following pair of curves:

a) $r = \sin \theta + \cos \theta, \quad r = 2 \sin \theta$

Ans: $\pi/4$

b) $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$

Ans: $\pi/3$.

c) $r = a$ and $r = 2a \cos \theta$.

Ans: $\pi/3$.

d) $r = \frac{a}{\log \theta}$ and $r = a \log \theta$

Ans: $\tan^{-1} \left(\frac{2e}{1-e^2} \right)$.

e) $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

Ans: $\tan^{-1} 3$.

f) $r = a$ and $r = 2a \cos \theta$.

Ans: $\pi/3$.

g) $r = 3 \cos(\theta)$ and $r = 1 + \cos(\theta)$

Ans: $\frac{\pi}{6}$.

5. Show that the following pair of curves intersect each other orthogonally.

a) $r = a(1 + \cos \theta)$ and $r = b(1 - \cos \theta)$.

b) $r = a \cos \theta$ and $r = a \sin \theta$.

c) $r = 4 \sec^2(\theta/2)$ and $r = 9 \operatorname{cosec}^2(\theta/2)$.

d) $r^n = a^n \cos(n\theta)$ and $r^n = b^n \sin(n\theta)$.

e) $r^2 \sin 2\theta = a^2$ and $r^2 \cos 2\theta = b^2$.

f) $r = ae^\theta$ and $re^\theta = b$.

g) $\frac{2a}{r} = 1 + \cos \theta$ and $\frac{2b}{r} = 1 - \cos \theta$.

h) $r = a \cos \theta$ and $r = a \sin \theta$.

i) $r = a\theta$ and $r = \frac{a}{\theta}$

Length of the perpendicular from pole to the tangent for the polar curve:

$$p = r \sin \phi \quad \text{or} \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

p is the length of perpendicular from pole to the tangent

ϕ is the angle between radius vector and tangent



**B.M.S. COLLEGE OF ENGINEERING
DEPARTMENT OF MATHEMATICS**

Dept. of Math., BMSCE

Unit 2: Calculus of One Variable

- If p denotes the length of the perpendicular from pole to the tangent of the curve $r = f(\theta)$, then prove that $p = r \sin \phi$ and hence deduce that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$
- Find the length of the perpendicular from the pole to the tangent for the following curves
 - $r = a(1 - \cos \theta)$ at $\theta = \pi/2$ **Ans:** $a/\sqrt{2}$
 - $r = a(1 + \cos \theta)$ at $\theta = \pi/2$. **Ans:** $a/\sqrt{2}$
 - $r^2 = a^2 \cos 2\theta$ at $\theta = \pi$. **Ans:** a
 - $r^2 = a^2 \sec 2\theta$ at $\theta = \pi/6$. **Ans:** $a/\sqrt{2}$.
 - $\frac{2a}{r} = (1 - \cos \theta)$ at $\theta = \pi/2$ **Ans:** $\sqrt{2}(a)$
 - $r = a \sec^2(\theta/2)$ at $\theta = \pi/3$ **Ans:** $2a/\sqrt{3}$

Pedal Equation:

Relation between r and p , obtained using $p = r \sin \phi$ or $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$

- Find the pedal equation of the following curves:
 - $r = ae^{\theta \cot \alpha}$. **Ans:** $p = r \sin \alpha$
 - $r(1 - \cos \theta) = 2a$ **Ans:** $p^2 = ar$
 - $r^m \cos(m\theta) = a^m$ **Ans:** $pr^{m-1} = a^m$.
 - $\frac{l}{r} = 1 + e \cos \theta$ **Ans:** $\frac{1}{p^2} = \frac{e^2 - 1}{1} + \frac{2}{r}$.

Curvature and Radius of Curvature

$$\text{Curvature} = \kappa = \frac{d\psi}{ds}.$$



**B.M.S. COLLEGE OF ENGINEERING
DEPARTMENT OF MATHEMATICS**

Dept. of Math., BMSCE

Unit 2: Calculus of One Variable

Radius of curvature = $\rho = \frac{1}{\kappa}$; $\kappa \neq 0$.

Cartesian Form:

Radius of curvature (Cartesian form), $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$, where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$

If $y_1 \rightarrow \infty$ then $\rho = \frac{\left[\left(\frac{dx}{dy}\right)^2 + 1\right]^{3/2}}{\frac{d^2x}{dy^2}}$

Find the radius of curvature for the following curves:

a. The Folium $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

b. Catenary $y = c \cosh\left(\frac{x}{c}\right)$ at $(0, c)$.

c. $y^2 = \frac{a^2(a-x)}{x}$ at $(a, 0)$.

Ans: $\frac{a}{2}$

d. $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$.

Ans: $\frac{a}{\sqrt{2}}$

e. $y = 4\sin x - \sin(2x)$ at $x = \frac{\pi}{2}$.

Ans: $\frac{5\sqrt{5}}{4}$

f. $r(1 + \cos \theta) = a$

Ans: $\frac{(2r)^{3/2}}{\sqrt{a}}$.

g. $r^n = a^n \cos n\theta$

Ans: $\frac{a^n}{(n+1)r^{n-1}}$.

h. $r = a(1 + \cos \theta)$

Ans: $\frac{2}{3}\sqrt{2ar}$.

2. Find the radius of curvature for $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a, 0)$ and $(0, b)$. Ans:

3. Find the radius of curvature for $y^2 = 4ax$ at (x, y) . Ans: $1 + \frac{y^2}{4a^2}$

4. If ρ is the radius of curvature for $y = \frac{ax}{a+x}$ then prove that $\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2 = \left(\frac{2\rho}{a}\right)^{2/3}$.



**B.M.S. COLLEGE OF ENGINEERING
DEPARTMENT OF MATHEMATICS**

Dept. of Math., BMSCE

Unit 2: Calculus of One Variable

Polar form:

Radius of curvature in polar coordinates: $\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$; where $r_1 = \frac{dr}{d\theta}$ and $r_2 = \frac{d^2r}{d\theta^2}$

Alternative:

Radius of curvature in **Pedal form**: $\rho = r \frac{dr}{dp}$

1. If ρ_1 and ρ_2 are the radii of curvature at the extremities of a chord through the pole for the polar curve $r = a(1 + \cos \theta)$, prove that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.
2. Show that for the curve $r(1 - \cos \theta) = 2a$, ρ^2 varies as r^3 .
3. For the cardioid $r = a(1 + \cos \theta)$, show that $\frac{\rho^2}{r}$ is constant.
4. Find the radius of curvature at the point (r, θ) for the curve $r^n = a^n \sin n\theta$.
5. Find the radius of curvature for the curve $\frac{l}{r} = 1 + e \cos \theta$ at any point (r, θ) .
6. Write the $p - r$ equation of the polar curve $r^n = a^n \sin n\theta$ and find the radius of curvature to the curve.
7. Find the pedal equation of the polar curve $r = f(\theta)$ and find the radius of curvature at any point (r, θ)

- | | | |
|-------|-------------------------------|--------------------------------------|
| (i) | $r = a(1 + \cos \theta)$ | Ans: $\frac{2}{3}\sqrt{2ar}$ |
| (ii) | $r = ae^{\theta \cot \alpha}$ | Ans: $r \operatorname{cosec} \alpha$ |
| (iii) | $r^2 = a^2 \cos 2\theta$ | Ans: $\frac{a^2}{3r}$ |
| (iv) | $r^m = a^m \cos m\theta$ | Ans: $\frac{a^m}{(m+1)r^{m-1}}$ |