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DEPARTMENT OF MATHEMATICS**

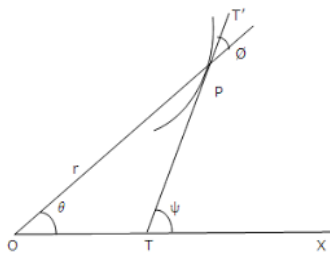
Course Code: 25MA1BSCM

Course: Mathematical Foundation for Civil, Electrical and Mechanical Streams– 1

UNIT-2: Calculus of One Variable

Polar Curves:

Angle between radius vector and tangent:



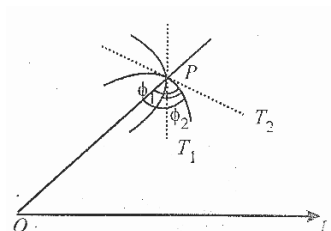
If $r = f(\theta)$, then the angle between radius vector and tangent is given by: $\tan \phi = r \frac{d\theta}{dr}$

1. If ϕ be the angle between radius vector and the tangent at any point of the curve $r = f(\theta)$ then prove that $\tan(\phi) = r \frac{d\theta}{dr}$.
2. Find the angle between the radius vector and the tangent for the following polar curves.
 - a) $r = a(1 + \cos \theta)$ **Ans:** $\pi/2 + \theta/2$.
 - b) $r^2 = a^2 \sin^2 \theta$ **Ans:** $\phi = \theta$
 - c) $\frac{l}{r} = 1 + e \cos \theta$ **Ans:** $\phi = \tan^{-1} \left[\frac{1 + e \cos \theta}{e \sin \theta} \right]$.
 - d) $r^m \cos m\theta = a^m$ **Ans:** $\pi/2 - m\theta$
3. Find the angle between the radius vector and the tangent for the following polar curves. And also find slope of the tangent at the given point.
 - e) $\frac{2a}{r} = 1 - \cos \theta$ at $\theta = 2\pi/3$ **Ans:** $\phi = \frac{2\pi}{3}, \tan \psi = \sqrt{3}$.
 - f) $r \cos^2(\theta/2) = a^2$ at $\theta = 2\pi/3$ **Ans:** $\phi = \frac{\pi}{6}$.
 - g) $r^2 \cos(2\theta) = a$ **Ans:** $\phi = \frac{\pi}{2} - 2\theta; \psi = \frac{\pi}{2} - \theta$



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Angle between curves:



Angle of intersection of two polar curves = angle of intersection of their tangents denoted by α where

$$\alpha = |\phi_2 - \phi_1| \quad \text{or} \quad \tan \alpha = \left| \frac{\tan \phi_1 - \tan \phi_2}{1 + \tan \phi_1 \tan \phi_2} \right|$$

1. Find the angle of intersection of the following pair of curves:

a) $r = \sin \theta + \cos \theta, \quad r = 2 \sin \theta$

Ans: $\pi/4$

b) $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$

Ans: $\pi/3$.

c) $r = a$ and $r = 2a \cos \theta$.

Ans: $\pi/3$.

d) $r = \frac{a}{\log \theta}$ and $r = a \log \theta$

Ans: $\tan^{-1} \left(\frac{2e}{1-e^2} \right) = 2 \tan^{-1}(e)$.

e) $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

Ans: $\tan^{-1} 3$.

f) $r = a$ and $r = 2a \cos \theta$.

Ans: $\pi/3$

g) $r = 3 \cos(\theta)$ and $r = 1 + \cos(\theta)$

Ans: $\frac{\pi}{6}$.

2. Show that the following pair of curves intersect each other orthogonally.

a) $r = a(1 + \cos \theta)$ and $r = b(1 - \cos \theta)$.

b) $r = a \cos \theta$ and $r = a \sin \theta$.

c) $r = 4 \sec^2(\theta/2)$ and $r = 9 \operatorname{cosec}^2(\theta/2)$.

d) $r^n = a^n \cos(n\theta)$ and $r^n = b^n \sin(n\theta)$.

e) $r^2 \sin 2\theta = a^2$ and $r^2 \cos 2\theta = b^2$.

f) $r = ae^\theta$ and $re^\theta = b$.

g) $\frac{2a}{r} = 1 + \cos \theta$ and $\frac{2b}{r} = 1 - \cos \theta$.

h) $r = a \cos \theta$ and $r = a \sin \theta$.

i) $r = a\theta$ and $r = \frac{a}{\theta}$



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Length of the perpendicular from pole to the tangent for the polar curve:

If p denotes the length of the perpendicular from the pole to the tangent of the polar curve $r = f(\theta)$, then $p = r \sin \phi$ and $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$, where ϕ is the angle between the radius vector and the tangent.

- If p denotes the length of the perpendicular from pole to the tangent of the curve $r = f(\theta)$, then prove that $p = r \sin \phi$ and hence deduce that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$
- Find the length of the perpendicular from the pole to the tangent for the following curves
 - $r = a(1 - \cos \theta)$ at $\theta = \pi/2$ **Ans:** $a/\sqrt{2}$
 - $r = a(1 + \cos \theta)$ at $\theta = \pi/2$. **Ans:** $a/\sqrt{2}$
 - $r^2 = a^2 \cos 2\theta$ at $\theta = \pi$. **Ans:** a
 - $r^2 = a^2 \sec 2\theta$ at $\theta = \pi/6$. **Ans:** $a/\sqrt{2}$.
 - $\frac{2a}{r} = (1 - \cos \theta)$ at $\theta = \pi/2$ **Ans:** $\sqrt{2}(a)$
 - $r = a \sec^2(\theta/2)$ at $\theta = \pi/3$ **Ans:** $2a/\sqrt{3}$

Pedal Equation:

If r be the radius vector at any point on a polar curve and p be the length of the perpendicular from the pole to the tangent at that point then the relation between r and p is called the pedal equation of the curve.

The pedal equation is obtained by using the equations $r = f(\theta)$, $p = r \sin \phi$ or $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$

- Find the pedal equation of the following curves:
 - $r = ae^{\theta \cot \alpha}$. **Ans:** $p = r \sin \alpha$
 - $r(1 - \cos \theta) = 2a$ **Ans:** $p^2 = ar$
 - $r^m \cos(m\theta) = a^m$ **Ans:** $pr^{m-1} = a^m$.
 - $\frac{l}{r} = 1 + e \cos \theta$ **Ans:** $\frac{1}{p^2} = \frac{e^2 - 1}{l^2} + \frac{2}{lr}$.



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- e) $r^m = a^m (\cos m\theta + \sin m\theta)$ Ans: $\sqrt{2}pa^m = r^{m+1}$.
- f) $a\theta = \sqrt{r^2 - a^2} - a \cos^{-1} \left(\frac{a}{r} \right)$ Ans: $p = \sqrt{r^2 - a^2}$.
- g) $r^n = a^n \cos n\theta$ Ans: $pr^{m-1} = a^m$
- h) $\frac{2a}{r} = (1 - \cos \theta)$ Ans: $p^2 = ar$

Curvature and Radius of Curvature

Curvature $= \kappa = \frac{d\psi}{ds}$. Radius of curvature $= \rho = \frac{1}{\kappa}$; $\kappa \neq 0$.

Cartesian Form: Radius of curvature (Cartesian form),

$$\rho = \left| \frac{(1 + y_1^2)^{3/2}}{y_2} \right|, \text{ where } y_1 = \frac{dy}{dx} \text{ and } y_2 = \frac{d^2y}{dx^2}$$

If $y_1 \rightarrow \infty$ then $\rho = \left| \frac{(1 + x_1^2)^{3/2}}{x_2} \right| = \left| \frac{1}{x_2} \right|$, where $x_1 = \frac{dx}{dy}$ and $x_2 = \frac{d^2x}{dy^2}$

1. Find the radius of curvature for the following curves:

Q. No	Curve	Solution
a	Folium of Descartes $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$	$\frac{3a}{8\sqrt{2}}$
b	Catenary $y = c \cosh\left(\frac{x}{c}\right)$ at $(0, c)$	$c \cos^2\left(\frac{x}{c}\right)$
c	$xy^2 = a^2(a - x)$ at $(a, 0)$.	$\frac{a}{2}$
d	$\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\frac{a}{4}, \frac{a}{4}\right)$	$\frac{a}{\sqrt{2}}$
e	$y = 4\sin x - \sin(2x)$ at $x = \frac{\pi}{2}$.	$\frac{5\sqrt{5}}{4}$
f	$xy^2 = a^3 - x^3$ at $(a, 0)$.	$\frac{3a}{2}$



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g	$y^2 = 4ax$ at (x, y)	$\frac{(y^2 + 4a^2)^{\frac{3}{2}}}{4a^2}$
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2. Find the radius of curvature for $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a, 0)$ and $(0, b)$. **Ans:** $\frac{a^2}{b}$
3. If ρ is the radius of curvature for $y = \frac{ax}{a+x}$ then prove that $\left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2 = \left(\frac{2\rho}{a}\right)^{\frac{2}{3}}$.
4. If ρ be the radius of curvature at any point P on the parabola $y^2 = 4ax$ and S be its focus then show that ρ^2 varies as SP^3 .

Polar form:

Radius of curvature in polar coordinates: $\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$; where $r_1 = \frac{dr}{d\theta}$ and $r_2 = \frac{d^2r}{d\theta^2}$

Alternative: Radius of curvature in **Pedal form:** $\rho = r \frac{dr}{dp}$

1. Find the radius of curvature at the point (r, θ) for the following polar curves:

Q.No	Curve	Solution
a	$r(1 + \cos \theta) = a$	$\frac{(2r)^{3/2}}{\sqrt{a}}$
b	$r^n = a^n \cos n\theta$	$\frac{a^n}{(n+1)r^{n-1}}$
c	$r = a(1 + \cos \theta)$	$\frac{2}{3} \sqrt{2ar}$
d	$r^n = a^n \sin n\theta$	$\frac{a^n}{(n+1)r^{n-1}}$
e	$\frac{l}{r} = 1 + e \cos \theta$	
f	$r = ae^{\theta \cot \alpha}$	$r \cos e c \alpha$
g	$r^2 = a^2 \cos 2\theta$	$\frac{a^2}{3r}$
h	$r(1 - \cos \theta) = a$	$\frac{(2r)^{3/2}}{\sqrt{a}}$



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2. If ρ_1 and ρ_2 are the radii of curvature at the extremities of a chord through the pole for the polar curve $r = a(1 + \cos \theta)$, prove that $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.
3. Show that for the curve $r(1 - \cos \theta) = 2a$, ρ^2 varies as r^3 .
4. For the cardioid $r = a(1 + \cos \theta)$, show that $\frac{\rho^2}{r}$ is constant.