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Polar Curves:
Angle between radius vector and tangent:

If $r = f(\theta)$, then the angle between radius vector and tangent is given by

$\tan\phi = r\dfrac{d\theta}{dr}$.

1. If ϕ be the angle between radius vector and the tangent at any point of the curve $r = f(\theta)$ then prove that $tan(\phi) = r\dfrac{d\theta}{dr}$.

2. Find the angle between the radius vector and the tangent for the following polar curves.

a) $r = a(1+\cos\theta)$ **Ans:** $\pi/2 + \theta/2$.

b) $r^2 = a^2\sin^2\theta$ **Ans:** $\phi = \theta$

c) $\dfrac{l}{r} = 1 + e\cos\theta$ **Ans:** $\phi = \tan^{-1}\left[\dfrac{1+e\cos\theta}{e\sin\theta}\right]$.

d) $r^m\cos m\theta = a^m$ **Ans:** $\pi/2 - m\theta$

3. Find the angle between the radius vector and the tangent for the following polar curves. And also find slope of the tangent at the given point.

e) $\dfrac{2a}{r} = 1 - \cos\theta$ at $\theta = 2\pi/3$ **Ans:** $\phi = \dfrac{2\pi}{3}, \tan\psi = \sqrt{3}$,.

f) $r\cos^2(\theta/2) = a^2$ at $\theta = 2\pi/3$ **Ans:** $\phi = \dfrac{\pi}{6}$.

g) $r^2\cos(2\theta) = a$ **Ans:** $\phi = \dfrac{\pi}{2} - 2\theta;\ \psi = \dfrac{\pi}{2} - \theta$

Angle between curves:

Angle of intersection of two polar curves = angle of intersection of their tangents denoted by α

$$\alpha = |\phi_2 - \phi_1| \text{ or } \tan\alpha = \left|\frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}\right|$$

4. Find the angle of intersection of the following pair of curves:

a) $r = sin\theta + cos\theta$, $r = 2sin\theta$ **Ans:** $\pi/4$

b) $r^2\sin 2\theta = 4$ and $r^2 = 16\sin 2\theta$ **Ans:** $\pi/3$.

c) $r = a$ and $r = 2a\cos\theta$. **Ans:** $\pi/3$.

d) $r=\dfrac{a}{\log\theta}$ and $r=a\log\theta$ **Ans**: $\tan^{-1}\left(\dfrac{2e}{1-e^2}\right)$.

e) $r=\dfrac{a\theta}{1+\theta}$ and $r=\dfrac{a}{1+\theta^2}$ **Ans**: $\tan^{-1}3$.

f) $r=a$ and $r=2a\cos\theta$. **Ans**: $\pi/3$.

g) $r=3\cos(\theta)$ and $r=1+\cos(\theta)$ **Ans**: $\dfrac{\pi}{6}$.

5. Show that the following pair of curves intersect each other orthogonally.

a) $r=a(1+\cos\theta)$ and $r=b(1-\cos\theta)$.

b) $r=a\cos\theta$ and $r=a\sin\theta$.

c) $r=4\sec^2(\theta/2)$ and $r=9\,\text{cosec}^2(\theta/2)$.

d) $r^n=a^n\cos(n\theta)$ and $r^n=b^n\sin(n\theta)$.

e) $r^2\sin 2\theta=a^2$ and $r^2\cos 2\theta=b^2$.

f) $r=ae^\theta$ and $re^\theta=b$.

g) $\dfrac{2a}{r}=1+\cos\theta$ and $\dfrac{2b}{r}=1-\cos\theta$.

h) $r=a\cos\theta$ and $r=a\sin\theta$.

i) $r=a\theta$ and $r=\dfrac{a}{\theta}$

Length of the perpendicular from pole to the tangent for the polar curve:

$p=r\sin\phi$ or $\dfrac{1}{p^2}=\dfrac{1}{r^2}+\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$

p is the length of perpendicular from pole to the tangent
ϕ is the angle between radius vector and tangent

1. If p denotes the length of the perpendicular from pole to the tangent of the curve $r = f(\theta)$, then prove that $p = r\sin\phi$ and hence deduce that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$
2. Find the length of the perpendicular from the pole to the tangent for the following curves
   a) $r = a(1-\cos\theta)$ at $\theta = \pi/2$ **Ans:** $a/\sqrt{2}$
   b) $r = a(1+\cos\theta)$ at $\theta = \pi/2$. **Ans:** $a/\sqrt{2}$
   c) $r^2 = a^2\cos 2\theta$ at $\theta = \pi$. **Ans:** a
   d) $r^2 = a^2\sec 2\theta$ at $\theta = \pi/6$. **Ans:** $a/\sqrt{2}$.
   e) $\frac{2a}{r} = (1-\cos\theta)$ at $\theta = \pi/2$ **Ans:** $\sqrt{2}(a)$
   f) $r = a\sec^2(\theta/2)$ at $\theta = \pi/3$ **Ans:** $2a/\sqrt{3}$

Pedal Equation:

Relation between r and p, obtained using $p = r\sin\phi$ or $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

1. Find the pedal equation of the following curves:
   a) $r = ae^{\theta\cot\alpha}$. **Ans:** $p = r\sin\alpha$
   b) $r(1-\cos\theta) = 2a$ **Ans**: $p^2 = ar$
   c) $r^m\cos(m\theta) = a^m$ **Ans**: $pr^{m-1} = a^m$.
   d) $\frac{l}{r} = 1 + e\cos\theta$ **Ans**: $\frac{1}{p^2} = \frac{e^2-1}{l^2} + \frac{2}{lr}$.

Curvature and Radius of Curvature

Curvature $= \kappa = \frac{d\psi}{ds}$.

Radius of curvature = $\rho = \frac{1}{\kappa}$; $\kappa \neq 0$.

Cartesian Form:

Radius of curvature (Cartesian form), $\rho = \dfrac{\left(1+y_1^2\right)^{3/2}}{y_2}$, where $y_1 = \dfrac{dy}{dx}$ and $y_2 = \dfrac{d^2y}{dx^2}$

If $y_1 \to \infty$ then $\rho = \dfrac{\left[\left(\frac{dx}{dy}\right)^2 + 1\right]^{3/2}}{\frac{d^2x}{dy^2}}$

Find the radius of curvature for the following curves:

a. The Folium $x^3 + y^3 = 3axy$ at the point $\left(3a/2, 3a/2\right)$

b. Catenary $y = c\cosh\left(x/c\right)$ at $(0, c)$.

c. $y^2 = \dfrac{a^2(a-x)}{x}$ at $(a, 0)$. Ans: $a/2$

d. $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $\left(\dfrac{a}{4}, \dfrac{a}{4}\right)$. Ans: $\dfrac{a}{\sqrt{2}}$

e. $y = 4\sin x - \sin(2x)$ at $x = \dfrac{\pi}{2}$. Ans: $\dfrac{5\sqrt{5}}{4}$

f. $r(1+\cos\theta) = a$ **Ans**: $\dfrac{(2r)^{3/2}}{\sqrt{a}}$.

g. $r^n = a^n \cos n\theta$ **Ans**: $\dfrac{a^n}{(n+1)r^{n-1}}$.

h. $r = a(1+\cos\theta)$ **Ans**: $\dfrac{2}{3}\sqrt{2ar}$.

2. Find the radius of curvature for $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at $(a, 0)$ and $(0, b)$. Ans:

3. Find the radius of curvature for $y^2 = 4ax$ at (x, y). Ans:

4. If ρ is the radius of curvature for $y = \dfrac{ax}{a+x}$ then prove that $\left(\dfrac{x}{y}\right)^2 + \left(\dfrac{y}{x}\right)^2 = \left(\dfrac{2\rho}{a}\right)^{2/3}$.

Polar form:

Radius of curvature in polar coordinates: $\rho = \dfrac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - rr_2}$; where $r_1 = \dfrac{dr}{d\theta}$ and $r_2 = \dfrac{d^2 r}{d\theta^2}$

Alternative:

Radius of curvature in **Pedal form**:

$$\rho = r\frac{dr}{dp}$$

1. If ρ_1 and ρ_2 are the radii of curvature at the extremities of a chord through the pole for the polar curve $r = a(1+\cos\theta)$, prove that $\rho_1^2 + \rho_2^2 = \dfrac{16a^2}{9}$.
2. Show that for the curve $r(1-\cos\theta) = 2a$, ρ^2 varies as r^3.
3. For the cardioid $r = a(1+\cos\theta)$, show that $\dfrac{\rho^2}{r}$ is constant.
4. Find the radius of curvature at the point (r,θ) for the curve $r^n = a^n \sin n\theta$.
5. Find the radius of curvature for the curve $\dfrac{l}{r} = 1 + e\cos\theta$ at any point (r,θ).
6. Write the $p-r$ equation of the polar curve $r^n = a^n \sin n\theta$ and find the radius of curvature to the curve.
7. Find the pedal equation of the polar curve $r = f(\theta)$ and find the radius of curvature at any point (r,θ)

(i)	$r = a(1+\cos\theta)$	Ans: $\dfrac{2}{3}\sqrt{2ar}$
(ii)	$r = ae^{\theta\cot\alpha}$	Ans: $r\,\mathrm{cosec}\,\alpha$
(iii)	$r^2 = a^2\cos 2\theta$	Ans: $\dfrac{a^2}{3r}$
(iv)	$r^m = a^m \cos m\theta$	Ans: $\dfrac{a^m}{(m+1)r^{m-1}}$

Parametric form:

Radius of curvature in parametric form:

$$\rho = \frac{\left(x_1^2 + y_1^2\right)^{3/2}}{x_1 y_2 - x_2 y_1}; \text{ where } x_1 = \frac{dx}{dt}; y_1 = \frac{dy}{dt}, x_2 = \frac{d^2x}{dt^2}, y_2 = \frac{d^2y}{dt^2}$$

Find the radius of curvature for the following curves:

1. $x = 6t^2 - 3t^4;\ y = 8t^3$

1. $x = e^t + e^{-t};\ y = e^t - e^{-t}$ at $t = 0$
2. $x = \frac{a\cos(t)}{t};\ y = \frac{a\sin(t)}{t}$
3. $x = a\ln(\sec t + \tan t);\ y = a\sec t$
4. $x = 1 - t^2;\ y = t - t^3;\ at\, t = \pm 1$
5. $x = 2t^2 - t^4;\ y = 4t^3\ at\, t = 1$
6. $x = a\left(t - \frac{t^3}{3}\right);\ y = at^2$
7. $x = \ln(t);\ y = \frac{1}{2}\left(t + t^{-1}\right)$
8. $x = a(t + \sin t);\ y = a(1 - \cos t)\ at\ t = \pi$
9. $x = a\cos t;\ y = a\sin t$
10. $x = a\ln\left(\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right);\ y = a\sec(\theta)$
11. $x = a\cos^3 t;\ y = a\sin^3 t$

Self Study Topics:

Center of Curvature:

A point C on the normal at any point P of a curve distant ρ from it, is called the center of curvature at P.

$\bar{x} = x - \frac{y_1\left(1+y_1^2\right)}{y_2}$ *and* $\bar{y} = y + \frac{\left(1+y_1^2\right)}{y_2}$

Equation of the circle of curvature:

$\left(x-\bar{x}\right)^2 + \left(y-\bar{y}\right)^2 = \rho^2$

1. Find the coordinates of the center of curvature at $\left(at^2, 2at\right)$ on the parabola $y^2 = 4ax$.
2. Find the coordinates of the center of curvature at any point of the parabola $y^2 = 4ax$.
3. Find the circle of curvature at the point $\left(a/4, a/4\right)$ of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$.
4. Find the circle of curvature at the point $\left(3/2, 3/2\right)$ of the curve $x^3 + y^3 = 3xy$.
5. Show that the circle of curvature at the origin for the curve $x + y = ax^2 + by^2 + ex^3$ is $\left(a+b\right)\left(x^2+y^2\right) = 2\left(x+y\right)$.

Evolute:

The locus of the center of curvature for a curve is called its evolute and the curve is called an involute.

1. Show that the equation of the evolute of the parabola $y^2 = 4ax$ is $27ay^2 = 4\left(x-2a\right)^3$.
2. Show that the evolute of the cycloid $x = a\left(\theta - \sin\theta\right), y = a\left(1-\cos\theta\right)$ is another equal cycloid.
3. Show that the equation of the evolute of the parabola $x^2 = 4ay$ is $4\left(y-2a\right)^3 = 27ax^2$.
4. Show that the equation of the evolute of the curve $x = a\left(\cos t + \log\tan(t/2)\right), y = a\sin t$ is $y = a\cosh\left(x/a\right)$.
5. Show that the equation of the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\left(ax\right)^{2/3} + \left(by\right)^{2/3} = \left(a^2 - b^2\right)^{2/3}$.
6. Find the evolute of the following:
   (i) Ellipse : $x = a\cos\theta, y = b\sin\theta$
   (ii) Cycloid: $x = a\left(t + \sin t\right), y = a\left(1-\cos t\right)$
   (iii) $x = a\cos^3 t,\ y = a\sin^3 t$

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