Differential equations:
Any equation which involves derivatives, dependent variables and independent variables is called a differential equation.

Classification	
Ordinary differential equations	Linear differential equations Nonlinear differential equations
Partial differential equations	Linear, semi linear, quasi linear …

Solution of a differential equation: A relation between x and y which satisfies the given differential equation.

General Solution: A solution which contains arbitrary constants.

Solution of first order ODE: Geometrically, the solution of a first order ODE represents a family of curves.

Linear Ordinary differential equations of 1st order

An equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$ is called a linear first order differential equation.

The solution is given by $y(IF)=\int(IF)Q+c$ where $IF=e^{\int Pdx}$ and c is the constant of integration.

I. BERNOULLI'S EQUATION

The equation of the form $\frac{dy}{dx}+P(x)y=Q(x)y^n$ is called the Bernoulli's equation. It can be reduced to a linear differential equation by dividing throughout by y^n, to get $y^{-n}\frac{dy}{dx}+Py^{1-n}=Q$ and substituting $y^{1-n}=z$.

EXAMPLES:

Solve the following differential equations:

1 $x\frac{dy}{dx}+y=x^3y^6.$

2 $y'+\frac{y}{2x}=\frac{x}{y^3}, y(1)=2.$

3 $(y\log x-2)ydx-xdy=0.$

4 $(x^3y^2+xy)dx=dy.$

5 $\cos x dy = y(\sin x - y)dx.$

6 $r\sin\theta - \cos\theta \dfrac{dr}{d\theta} = r^2.$

7 $y^2 y' - y^3 \tan x - \sin x \cos^2 x = 0.$

8 $\dfrac{dy}{dx} = \dfrac{x^2 + y^2 + 1}{2xy}.$

9 $3y' + xy = xy^{-2}.$

10 $\dfrac{dy}{dx} = \dfrac{y}{x + \sqrt{xy}}.$

11 $\dfrac{dz}{dx} + \dfrac{z}{x}\log z = \dfrac{z}{x}(\log z)^2.$

II. EXACT DIFFERENTIAL EQUATIONS

A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if $M(x,y)dx + N(x,y)dy$ is the exact differential of some function $u(x,y)$ i.e. $du = Mdx + Ndy$. Its solution, therefore is $u(x,y) = c$.

The necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, and its solution is given by $\displaystyle\int_{(y\ \text{constant})} M dx + \int (\text{terms of } N \text{ without } x) dy = c$.

EXAMPLES:

Solve the following differential equations:

1 $\left(x^4 - 2xy^2 + y^4\right)dx - \left(2x^2y - 4xy^3 + \sin y\right)dy = 0.$

2 $\left[y\left(1 + \dfrac{1}{x}\right) + \cos y\right]dx + \left(x + \log x - x\sin y\right)dy = 0.$

3 $\left(2xy + y - \tan y\right)dx + \left(x^2 - x\tan^2 y + \sec^2 y\right)dy = 0.$

4 $\dfrac{2x}{y^3}dx + \dfrac{y^2 - 3x^2}{y^4}dy = 0.$

5 $y' = \dfrac{y - 2x}{2y - x}$, $y(1) = 2.$

6 $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0.$

7 $\left(\sec x \tan x \tan y - e^{x}\right)dx + \sec x \sec^{2} y dy = 0$.

8 $\left(y \sin 2x\right)dx - \left(1 + y^{2} + \cos^{2} x\right)dy = 0$.

9 $\left(\cos x - x\cos y\right)dy - \left(\sin y + y\sin x\right)dx = 0$.

10 $\left(\frac{y}{(x+y)^{2}} - 1\right)dx + \left(1 - \frac{x}{(x+y)^{2}}\right)dy = 0$.

11 $\sin x \cosh y dx - \cos x \sinh y dy = 0,\ y(0) = 3$.

III. DIFFERENTIAL EQUATIONS REDUCIBLE TO THE EXACT FORM

Type 1: In the differential equation $Mdx + Ndy = 0$, if $\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{N}$ a function of x say $f(x)$, then $IF = e^{\int f(x)dx}$.

<u>EXAMPLES:</u>

Solve the following differential equations by reducing it to exact differential equations

1 $\left(xy^{2} - e^{1/x^{3}}\right)dx - x^{2} y dy = 0.$

2 $y(x+y)dx + (x + 2y - 1)dy = 0.$

3 $(2y^{2} + 3x)dx + 2xydy = 0$.

4 $(x^{2} + y^{2} - 5)dx = (y + xy)dy,\ \ y(0) = 1$.

Type 2: In the differential equation $Mdx + Ndy = 0$, if $\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}{-M}$ a function of y say $F(y)$, then $IF = e^{\int F(y)dy}$.

<u>EXAMPLES:</u>

Solve the following differential equations by reducing it to exact differential equations

1 $\left(3x^{2}y^{4} + 2xy\right)dx + \left(2x^{3}y^{3} - x^{2}\right)dy = 0.$

2 $\left(xy^{3} + y\right)dx + 2\left(x^{2}y^{2} + x + y^{4}\right)dy = 0.$

3 $3\left(x^{2} + y^{2}\right)dx + x\left(x^{2} + 3y^{2} + 6y\right)dy = 0.$

4 $6xydx+(4y+9x^2)dy=0.$

5 $xdx+\left(x^2y+4y\right)dy=0,\ y\left(4\right)=0.$

APPLICATIONS

ORTHOGONAL TRAJECTORIES

In Cartesian Co-ordinates:

EXAMPLES:

Find the orthogonal trajectories of the family of:

a) $y=c\left(\sec x+\tan x\right).$

b) $y^2=4ax.$

c) $x^2-y^2=cx.$

d) $ay^2=x^3.$

e) Find the orthogonal trajectory of the family of confocal conics $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2+\lambda}=1,\lambda$ being the parameter.

f) Show that the family of curves are self-orthogonal:

a) Confocal conics, $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1,\lambda$ being the parameter.

b) $y^2=4a\left(x+a\right).$

c) $y^2=2cx+c^2.$

In Polar coordinates:

EXAMPLES:

Find the orthogonal trajectories (O.T.) of each of the following family of curves:

a) $r=\dfrac{2a}{\left(1+\cos\theta\right)}$

b) $r^2=a^2\cos 2\theta$

c) $r=4a\sec\theta\cdot\tan\theta$

d) $r = a\left(1+\sin^2\theta\right)$

e) $r = a\sin\theta\tan\theta$

f) $r = a\left(\sec\theta + \tan\theta\right)$

g) $r^n \cos n\theta = a^n$

NON-LINEAR DIFFERENTIAL EQUATIONS

Equation solvable for p :

The Equations of the form $f(x, y, p) = 0$.

WORKING RULE:

A differential equation of the first order but of the n^{th} degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots\dots + P_n = 0$$

where $P_1, P_2, \dots\dots, P_n$ are functions of x and y.

Splitting up the left hand side of the above equation into a P_1 linear factors, we have

$$[p - f_1(x, y)]\,[p - f_2(x, y)]\,\dots\dots[p - f_n(x, y)] = 0.$$

Equating each of the factor to zero,

$$p = f_1(x, y),\ p = f_2(x, y),\ \dots\dots p = f_n(x, y)$$

Solving each of these equations of the first order and first order and first degree, we get the solutions

$$F_1(x, y, c) = 0.\ F_2(x, y, c) = 0,\ \dots\dots F_n(x, y, c) = 0.$$

EXAMPLES:

Solve the following differential equations:

1 $\dfrac{dy}{dx} - \dfrac{dx}{dy} = \dfrac{x}{y} - \dfrac{y}{x}$.

2. $p^2 + 2py\cot x = y^2$.

3. $y\left(\dfrac{dy}{dx}\right)^2 + (x - y)\dfrac{dy}{dx} - x = 0$.

4. $p(p + y) = x(x + y)$.

5. $xy\left(\frac{dy}{dx}\right)^2-(x^2+y^2)\frac{dy}{dx}+xy=0$.
6. $y=x\left(p+\sqrt{1+p^2}\right)$.
7. $p^3+2xp^2-y^2p^2-2xy^2p=0$.
8. $x^2\left(\frac{dy}{dx}\right)^2+xy\frac{dy}{dx}-6y^2=0$.
9. $2p^3-(2x+4\sin x-\cos x)p^2-(x\cos x-4x\sin x+\sin 2x)p+x\sin 2x=0$.
10. $p^2-5p+6=0$.
11. $4y^2p^2+2xp(3x+1)+3x^2=0$.
12. $p-\frac{1}{p}-\frac{x}{y}+\frac{y}{x}=0$.
13. $xyp^2+p(3x^2-2y^2)-6xy=0$.
14. $p^3-(y+2x-e^{x-y})p^2+(2xy-2xe^{x-y}-ye^{x-y})p+2xye^{x-y}=0$.
15. $yp^2-(x-y)p-x=0$.
16. $xyp^2+(x^2+xy+y^2)p+x^2+xy=0$.

Clairaut's Equation

An equation of the form $y=px+f(p)$ is known as Clairaut's equation.

Working rule:

Differentiating with respect to x, we have $p=p+x\frac{dp}{dx}+f'(p)\frac{dp}{dx}$

or $[x+f'(p)]\frac{dp}{dx}=0$

$\therefore \frac{dp}{dx}=0$, or $x+f'(p)=0$

$\frac{dp}{dx}=0$, gives $p=c$

Eliminate p in the above equations, we get $y=cx+f(c)$ is the general solution.

Hence the solution of the Clairaut's equation is obtained on replacing p by c.

To obtain singular solution, we proceed as follows:

1. Find the general solution by replacing p by c.
2. Differentiate this w.r.t. c giving $x+f(c)=0$.

3. Eliminate c from the equations, which will be the singular solution.

EXAMPLES:

Find the general and singular solutions of the equations:

1. $p = \sin(y - xp)$.
2. $xp^2 - yp + a = 0$.
3. $p = \log(px - y)$.
4. $y = px + \sqrt{a^2 p^2 + b^2}$.
5. $\sin px \cos y = \cos px \sin y + p$.
6. $p = \ln(px - y)$.
7. $y = 4xp - 16y^3 p^2$.
8. $\sin y \cos^2 x = \cos^2 y p^2 + \sin x \cos x \cos y p$.
9. $(p-1)e^{3x} + p^3 e^{2y} = 0$.
10. $y = xp + \dfrac{ap}{\sqrt{1+p^2}}$.
11. $(x^2 - 1)p^2 - 2xyp + y^2 - 1 = 0$.
12. $p^2 y + px^3 - x^3 y = 0$.
13. $y = px + 2p^2$.
14. $a^2 p = (py + x)/(px - y)$.
15. $y + y^2 + p(2y - 2xy - x + 2) + xp^2(x-2) = 0$.
16. $(px^2 + y^2)(px + y) = (p+1)^2$.

Equations reducible to Clairaut's form

Solve the following equations:

1. $(px - y)(py + x) = a^2 p$.
2. $y + 2\left(\dfrac{dy}{dx}\right)^2 = (x+1)\dfrac{dy}{dx}$.
3. $(x-a)\left(\dfrac{dy}{dx}\right)^2 + (x-y)\dfrac{dy}{dx} - y = 0$.
4. $x^2(y - px) = yp^2$.
5. $(y - px)(p-1) = p$.
6. $(px + y)^2 = py^2$.
7. $(px - y)(x + py) = 2p$.

SELF-STUDY:

Applications of ODE's: L-R circuits:

1. A generator having emf 100 $volts$ is connected in series with a 10 ohm resistor and an inductor of 2 $henries$. If the switch is closed at a time $t=0$, determine the current time $t>0$.
2. Find the $I(t)$ in the RL – circuit with $R=10\Omega$, $L=100H$ and $E=40V$ when $0\le t\le 100$ and $E=0$ when $t>100$ and $I(0)=4$.
3. When a switch is closed in a circuit containing a battery E, a resistance R and an inductance L, the current build up a rate given by $L\frac{di}{dt}+Ri=E$. Find i as a function of t. How long will it be, before the current has reached one-half its initial value if $E=6V$, $R=100\Omega$ and $L=0.1H$?
4. Obtain the differential equation for the current i in an electrical circuit containing an inductance L and a resistor R in series and acted by an electromotive force $E\sin(\omega t)$. Find the value of the current at any time t, if initially there is no current in the circuit.
5. Find $I(t)$ in an RL – circuit with $E=10V$, $R=5\Omega$, $L=(10-t)H$ when $0\le t\le 10s$ and $L=0$ when $t>10s$ and $I(0)=0$. A capacitor $c=0.01F$ in series with a resistor $R=20\Omega$ is charged from a battery $E_0=10V$.

Equations solvable for x:

Solve the following equations:

1. $y=2px+y^2p^3$.
2. $p^3-4xyp+8y^2=0$.
3. $x-yp=ap^2$.
4. $p^3y+2px=y$.
5. $p=\tan\left(x-\frac{p}{1+p^2}\right)$.
6. $p^2-xp+y=0$.

Equations solvable for y:

Solve the following equations:

1. $y-2px=\tan^{-1}(xp^2)$.
2. $y=2px+p^n$.
3. $y=x+a\tan^{-1}p$.
4. $y+px=x^4p^2$.

5. $xp^2 + x = 2yp$.
6. $x^2\left(\frac{dy}{dx}\right)^4 + 2x\frac{dy}{dx} - y = 0$.
7. $y = p\sin p + \cos p$.
8. $3x^4 p^2 - xp - y = 0$.