UNIT 5

ORDINARY DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Solve the following Homogeneous Linear Differential equations with constant coefficients:

1. $\frac{d^2y}{dx^2}-\frac{dy}{dx}-12y=0$ **Ans:** $y=c_1e^{-3x}+c_2e^{4x}$
2. $y'''-4y''+y'+6y=0$ **Ans:** $y=c_1e^{-x}+c_2e^{2x}+c_3e^{3x}$
3. $\left(D^2-4D+1\right)y=0$ **Ans:** $y=c_1e^{\left(2+\sqrt{3}\right)x}+c_2e^{\left(2-\sqrt{3}\right)x}$
4. $y'''-6y''+11y'-6y=0,\ y(0)=0,\ y'(0)=0, y''(0)=2.$ **Ans:** $y=e^x-2e^{2x}+e^{3x}$
5. $y''-4y'+4y=0,\ y(0)=3, y'(0)=1$ **Ans:** $y=(3-5x)e^{2x}$
6. $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+3\frac{dy}{dx}-y=0$ **Ans:** $y=(c_1+c_2x+c_3x^2)e^x$
7. $\left(D^2+1\right)^3y=0$ **Ans:** $y=(c_1+c_2x+c_3x^2)\cos x+(c_4+c_5x+c_6x^2)\sin x$
8. $\frac{d^3x}{dt^3}+x=0$ **Ans:** $x=c_1e^{-t}+\left(c_2\cos\frac{\sqrt{3}t}{2}+c_3\sin\frac{\sqrt{3}t}{2}\right)e^{t/2}$
9. $y''-2y'+10y=0,\ y(0)=4,\ y'(0)=1.$ **Ans:** $y=e^x\left(4\cos3x-\sin3x\right)$

Solve the following non-Homogeneous Linear Differential equations with constant coefficients:

Case I: When $X=e^{ax}$

1. $\left(D^2+4\right)y=e^{3x}$ **Ans:** $y=\left(c_1e^{2ix}+c_2e^{-2ix}\right)+\frac{1}{13}e^{3x}$
2. $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=6e^{3x}+7e^{-2x}-\log2+3^x$

   Ans: $y=\left(c_1+c_2x\right)e^{3x}+3x^2e^{3x}+\frac{7}{25}e^{-2x}-\frac{1}{9}\log2+\frac{3^x}{\left(\log_e3\right)^2-6\left(\log_e3\right)+9}$
3. $\left(D^2-D+1\right)y=\sinh x$ **Ans:** $y=\left(c_1\cos\frac{\sqrt{3}x}{2}+c_2\sin\frac{\sqrt{3}x}{2}\right)e^{x/2}+\frac{1}{6}\left(3e^x-e^{-x}\right)$
4. $D\left(D+1\right)^2y=12e^{-x}$ **Ans:** $y=c_1+\left(c_2+c_3x\right)e^{-x}-6x^2e^{-x}$
5. $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=-2\cosh x,\quad y=0,\ \frac{dy}{dx}=1\ \text{ at }\ x=0$

   Ans: $y=\frac{3}{5}e^{-2x}\left(\cos x+3\sin x\right)-\frac{e^x}{10}-\frac{e^{-x}}{2}$

Problems on Case II : $X = \cos ax \ \ or \ \ \sin ax$

1. $\left(D^2+4\right)y = \sin 3x + \cos 2x$. **Ans:** $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{5}\sin 3x + \frac{x}{4}\sin 2x$

2. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{2x} - \cos^2 x$ **Ans:** $y = \left(c_1 + c_2 x\right)e^{-x} + \frac{e^{2x}}{9} - \frac{1}{2} - \frac{2}{25}sin\,2x + \frac{3}{50}cos\,2x$

3. $\left(D^2+5D-6\right)y = \sin 4x \sin x$

Ans: $y = c_1 e^{x} + c_2 e^{-6x} + \frac{\sin 3x - \cos 3x}{60} + \frac{31\cos 5x - 25\sin 5x}{3172}$

4. $y'' + 4y' + 4y = 3\sin x + 4\cos x,\ \ y(0) = 1 \text{ and } y'(0) = 0$ **Ans:** $y = (1+x)e^{-2x} + \sin x$

5. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x} + \sin 2x$

Ans: $y = c_1 + \left(c_2 + c_3 x\right)e^{-x} - \frac{x^2 e^{-x}}{2} + \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x$

Problems on Case III: $X = x^m$

1. $\left(D^2-1\right)y = 2x^4 - 3x + 1$ **Ans:** $y = c_1 e^{x} + c_2 e^{-x} - \left[2x^4 + 24x^2 - 3x + 49\right]$

2. $\left(D^6 - D^4\right)y = x^2$

Ans: $y = \left(c_1 + c_2 x + c_3 x^2 + c_4 x^3\right) + c_5 e^{x} + c_6 e^{-x} - \left[\frac{x^6}{360} + \frac{x^4}{12} + x^2 + 2\right]$

3. $\left(D^2+4D+4\right)y = x^2 + 2x,\ y(0) = 0,\ \ y'(0) = 0$ **Ans:** $y = -\frac{1}{8}(1+2x)e^{-2x} + \frac{1}{8}\left(2x^2 - 1\right)$

4. $(D-2)^2\, y = 8x^2$ **Ans:** $y = (c_1 + c_2 x)e^{2x} + 2x^2 + 4x + 3$

5. $\left(D^2+2\right)y = x^3 + e^{-2x} + \cos 3x$

Ans: $y = (c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + \frac{1}{2}\left[x^3 - 3x\right] + \frac{1}{6}e^{-2x} - \frac{1}{7}\cos 3x$

Method of variation of parameters

1. $\frac{d^2y}{dx^2}+y=\sec x$ **Ans:** $y=c_1\cos x+c_2\sin x+\cos x\log(\cos x)+x\sin x$

2. $\frac{d^2y}{dx^2}+y=\frac{1}{1+\sin x}$ **Ans:** $y=c_1\cos x+c_2\sin x+\sin x\log(1+\sin x)-x\cos x-1+\sin x$

3. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}=e^x\sin x$ **Ans:** $y=c_1+c_2e^{2x}-\frac{1}{2}e^x\sin x$

4. $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=\frac{1}{1+e^{-x}}$

Ans: $y=\left(e^x+e^{2x}\right)\log\left(1+e^x\right)+\left(c_1-1-x\right)e^x+\left(c_2-x-1\right)e^{2x}$

5. $\left(D^2+2D+1\right)y=e^{-x}\log x$.

Ans: $y=c_1e^{-x}+c_2xe^{-x}+\frac{x^2}{2}\left(\frac{1}{2}-\log x\right)e^{-x}+xe^{-x}\left(x\log x-x\right)$

6. $\frac{d^2y}{dx^2}-y=\frac{2}{\left(1+e^x\right)}$ **Ans:** $y=c_1e^x+c_2e^{-x}-1+e^x\log(e^{-x}+1)-e^{-x}\log\left(e^x+1\right)$

7. $y''-6y'+9y=\frac{e^{3x}}{x^2}$ **Ans:** $y=\left(c_1+c_2x\right)e^{3x}-e^{3x}\left(1+\log x\right)$

8. $y''-2y'+2y=e^x\tan x$ **Ans:** $y=\left(c_1\cos x+c_2\sin x\right)e^x-e^x\cos x\log\left(\sec x+\tan x\right)$

9. $\frac{d^2y}{dx^2}+a^2y=\cos ecax$ **Ans:** $y=\left(c_1-\frac{x}{a}\right)\cos ax+\left[c_2+\left(1/a^2\right)\log\sin ax\right]\sin ax$

10. $\left(D^2-2D+1\right)y=e^x\log x$ **Ans:** $y=\left(c_1+c_2x\right)e^x+\frac{x^2}{4}e^x\left(2\log x-3\right)$

LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Cauchy's linear differential equation:

1. $x\frac{d^2y}{dx^2}-\frac{2y}{x}=x+\frac{1}{x^2}$ **Ans:** $y=c_1x^2+\frac{c_2}{x}+\frac{1}{3}\left(x^2-\frac{1}{x}\right)\log x$

2. $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y=\frac{1}{1-x^2}$ **Ans:** $y=\left(c_1+c_2\log x\right)\frac{1}{x}+1-\frac{1}{x^2}$

3. $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x$ **Ans:** $y=\frac{c_1}{x}+\frac{\left(c_2+e^x\right)}{x^2}$

4. $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+6y=4x-6$ **Ans:** $y=c_1x^2+c_2x^3+2x-1$

Legendre's linear differential equation:

1. $(2x+3)^2\frac{d^2y}{dx^2}-2(2x+3)\frac{dy}{dx}-12y=6x$

Ans: $y=\frac{c_1}{(2x+3)}+c_2(2x+3)^3-\frac{3}{16}(2x+3)+\frac{3}{4}$

2. $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=\sin[2\log(1+x)]$

Ans: $y=c_1\cos\log(1+x)+c_2\sin\log(1+x)-\frac{1}{3}\sin[2\log(1+x)]$

3. $(x-1)^3\frac{d^3y}{dx^3}+2(x-1)^2\frac{d^2y}{dx^2}-4(x-1)\frac{dy}{dx}+4y=4\log(x-1)]$

Ans: $y=c_1(x-1)+c_2(x-1)^2+c_3(x-1)^{-2}+\log(x-1)+1$

4. $(x+2)^2y''+3(x+2)y'-3y=0$ **Ans:** $y=c_1(x+2)+c_2(x+2)^{-3}$

5. $(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1$

Ans: $y=c_1(3x+2)^{1/3}+c_2(3x+2)^{-1}+\frac{1}{27}\left[\frac{1}{15}(3x+2)^2+\frac{1}{4}(3x+2)-7\right]$

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