INTEGRALS CALCULUS

1. DOUBLE INTEGRALS

I. Evaluate the following:-
(Illustrative Examples)

1. $\int_0^1\int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}$.
2. $\int_1^2\int_3^4 (xy+e^y)dy\,dx$.
3. $\int_3^4\int_1^2 \frac{dy\,dx}{(x+y)^2}$.
4. $\int_1^2\int_0^x \frac{dydx}{x^2+y^2}$.
5. $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$.
6. $\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx$.
7. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}dxdy$.
8. $\int_0^{\pi}\int_0^{a(1-\cos\theta)} r\sin\theta\,drd\theta$.
9. $\int_0^{\pi}\int_0^{\sin\theta} r\,drd\theta$.

II. Evaluate the following over the specified region:-

1. $\iint_R x\,y\,dx\,dy$, where R is the region bounded by the circle $x^2+y^2=a^2$ in the first quadrant.
2. $\iint_A x\,y\,(x+y)\,dA$, where A is the area bounded by the parabola $y=x^2$ and the line $y=x$.

3. $\iint_R x\, y\, dx\, dy$, where R is the domain bounded by $x-axis,$ ordinate $x=2a$ and the curve $x^2=4ay$.
4. $\iint_D x^2 dx\, dy$, where D is the domain in the first quadrant bounded by the hyperbola $xy=16$, and the lines $y=x$, $y=0$ and $x=8$.
5. $\int\int r\sin\theta\, dr\, d\theta$ over the cardioid $r=a(1-\cos\theta)$ above the initial line.
6. $\iint_R r^2 \sin\theta\, dr\, d\theta$, where R is the semicircle $r=2a\cos\theta$ above the initial line.
7. $\int\int \frac{rdrd\theta}{\sqrt{a^2+r^2}}$ over one loop of the lemniscate $r^2=a^2\cos 2\theta$.
8. $\int\int r^3\, dr\, d\theta$ over the area bounded between the circles $r=2\cos\theta$ and $r=4\cos\theta$.
9. $\int\int r^3\, dr\, d\theta$ over the area bounded between the circles $r=2\sin\theta$ and $r=4\sin\theta$.

III. <u>Change the order of integration and hence evaluate the following:-</u>

1. $\int_0^a \int_y^a \frac{xdxdy}{x^2+y^2}.$
2. $\int_0^1 \int_x^1 \frac{1}{1+y^4}\, dy\, dx.$
3. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dydx$
4. $\int_0^\infty \int_0^x x\, e^{-x^2/y}\, dydx.$
5. $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy.$
6. $\int_0^1 \int_{e^x}^e \frac{dydx}{\log y}.$
7. $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx.$
8. $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2 dy\, dx}{\sqrt{y^4-a^2x^2}}.$
9. $\int_0^1 \int_x^{\sqrt{x}} xydydx.$

10. $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2)\, dx\, dy.$

11. $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} dy\, dx.$

12. $\int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy.$

13. $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx.$

IV. Evaluate the following by transforming into polar coordinates:-

1. $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2 + y^2)\, dy\, dx.$

2. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy.$

3. $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$

4. $\int_0^1 \int_0^{\sqrt{1-y^2}} e^{x^2+y^2}\, dx\, dy.$

5. $\int_0^2 \int_x^{\sqrt{8-x^2}} \frac{1}{5+x^2+y^2}\, dy\, dx.$

6. $\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{1+\sqrt{x^2+y^2}}\, dx\, dy.$

7. $\int_0^a \int_y^a \frac{x\, dx\, dy}{\sqrt{x^2+y^2}}.$

8. $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} dy\, dx.$

9. $\int_0^{\frac{a}{\sqrt{2}}} \int_y^{\sqrt{a^2-y^2}} \log_e (x^2 + y^2)\, dxdy$, where $(a>0)$

V. Area of the region using double integral

1. Find the area lying inside the cardioid $r=a(1+\cos\theta)$ and outside the circle $r=a$.
2. Find the area of Lemniscate $r^2=a^2\cos 2\theta$.
3. Find the area of the cardioid $r=a(1+\cos\theta)$.
4. Find the area which is inside the circle $r=3a\cos\theta$ and outside the cardioid $r=a(1+\cos\theta)$
5. Find the area lying inside the circle $r=a\sin\theta$ and outside the cardioid $r=a(1-\cos\theta)$.

2. TRIPLE INTEGRALS

VI. Evaluate the following:-

1. $\int_0^1\int_0^2\int_1^2 x^2yz\,dxdydz.$
2. $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}\left(x^2+y^2+z^2\right)dzdydx.$
3. $\int_0^2\int_1^z\int_0^{yz} xyzdxdydz.$
4. $\int_0^6\int_0^{6-x}\int_1^{6-x-z} dy\,dz\,dx.$
5. $\int_0^4\int_0^{\frac{1}{2}}\int_0^{x^2}\frac{1}{\sqrt{x^2-y^2}}dy\,dx\,dz.$
6. $\int_0^a\int_0^{\sqrt{a^2-z^2}}\int_0^{\sqrt{a^2-y^2-z^2}} xdxdydz.$
7. $\int_0^1\int_0^x\int_0^{x+y}(x+y+z)\,dz\,dy\,dx.$
8. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyzdzdydx.$
9. $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}.$
10. $\int_{-1}^{1}\int_0^z\int_{x-z}^{x+z}(x+y+z)\,dxdydz.$

11. $\int_0^1\int_0^1\int_0^{2-x^2-y^2} xye^z\, dz\, dx\, dy.$

12. $\int_0^4\int_0^{2\sqrt{z}}\int_0^{\sqrt{4z-x^2}} dydxdz.$

VII. Evaluate the following over the region R bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1.$

1. $\iiint_R (x+y+z)\,dx\,dy\,dz.$
2. $\int\int_R\int xyzdxdydz.$
3. $\iiint_R \frac{dx\,dy\,dz}{(1+x+y+z)^3}.$

VIII. Volume of the region using triple integral

1. Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the co-ordinate planes.
2. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$
3. Find the volume of the sphere $x^2+y^2+z^2=a^2.$

3. BETA AND GAMMA FUNCTIONS

I. Express the following in terms of Gamma functions:-

1. $\int_0^\infty e^{-kx}x^{p-1}dx,\ k>0.$
2. $\int_0^\infty \frac{x^4}{4^x}dx.$
3. $\int_0^\infty 3^{-4x^2}dx.$
4. $\int_0^1 x^m[\log_e x]^n dx$, where n is an integer and $m>-1.$
5. $\int_0^1 x^{q-1}\left[\log_e\left(\frac{1}{x}\right)\right]^{p-1}dx \quad (p>0, q>0).$
6. $\int_0^\infty e^{-t^2}t^{2n-1}dt.$

II. Express the following in terms of Gamma functions:-

1. $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta.$
2. $\int_0^{\pi/2} \left(\sqrt{\tan\theta} + \sqrt{\sec\theta}\right) d\theta.$
3. $\int_0^1 x^m \left(1-x^n\right)^p dx.$
4. $\int_0^1 \frac{dx}{\sqrt{1-x^4}}.$

III. Prove the following:-

1. $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$, given $\int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$ and hence deduce the value of $\int_0^\infty \frac{dy}{1+y^4}$.
2. $\int_0^\infty xe^{-x^8} dx \times \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}.$
3. $\int_0^{\pi/2} \sqrt{\sin\theta} d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi.$
4. $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}.$
5. $\frac{\beta(m+1,n)}{m} = \frac{\beta(m,n)}{m+n} = \frac{\beta(m,n+1)}{n}.$