I. PARTIAL DERIVATIVES

Let $u = f(x, y)$ be a function of two variables x and y. If we keep y as constant and vary x alone, then u is a function of x only. The derivative of u with respect to x, treating y as constant is called the partial derivative of u w. r. t x and is denoted by one of the symbols $\frac{\partial u}{\partial x}$, u_x, Thus $\frac{\partial u}{\partial x} = u_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$. Similarly $\frac{\partial u}{\partial y} = u_y = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$
Further u_x and u_y are also functions of x and y, so these can further be differentiated partially w. r. t. x and y. Thus $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$ or u_{xx}, $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x \partial y}$ or u_{yx}, $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x}$ or u_{xy} and $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$ or u_{yy}.

Examples

1. Prove that $y v_y - x v_x = -y^2 v^3$ if $v = \left(1 - 2xy + y^2\right)^{-1/2}$.
2. Show that $w_x + w_y + w_z = 0$ *if* $w = (y-z)(z-x)(x-y)$.
3. Show that $u_x + u_y = u$, if $u = \dfrac{e^{x+y}}{e^x + e^y}$.
4. If $w = x^2 y + y^2 z + z^2 x$, prove that $w_x + w_y + w_z = (x+y+z)^2$.
5. If $u = \dfrac{y}{z} + \dfrac{z}{x}$, show that $xu_x + yu_y + zu_z = 0$.
6. Given $u = e^{r\cos\theta}\cos(r\sin\theta)$, $v = e^{r\cos\theta}\sin(r\sin\theta)$, Prove that

   $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$.
7. If $(x+y)z = x^2 + y^2$, Show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right]$.
8. Show that $(v_x)^2 + (v_y)^2 + (v_z)^2 = v^4$ if $v = (x^2 + y^2 + z^2)^{-1/2}$.
9. If $z = e^{ax+by} f(ax - by)$ prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$.

10. If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, then show that

(i) $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$ and (ii) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

11. If $u = \log_e\left(x^3 + y^3 - x^2 y - xy^2\right)$, then show that $u_{xx} + 2u_{xy} + u_{yy} = -\frac{4}{(x+y)^2}$.

12. If $z = f(x+ct) + \phi(x-ct)$, then prove that $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$.

13. If $u = e^{a\theta}\cos(a\log_e r)$ Show that $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$.

14. Prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, if

a) $u = x^y$

b) $u = \log_e\left(\frac{x^2 + y^2}{xy}\right)$

15. Verify that $f_{xy} = f_{yx}$ when $f(x,y) = \sin^{-1}(y/x)$.

16. Prove that $u_{xx} + u_{yy} = 0$, if

a) $u = \tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)$

b) $u = \log_e\left(x^2 + y^2\right) + \tan^{-1}(y/x)$

17. Find the value of n so that the equation $v = r^n\left(3\cos^2\theta - 1\right)$ satisfies the relation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) = 0.$$

18. If $Z = \tan(y + ax) + (y - ax)^{3/2}$, Show that $Z_{xx} = a^2 Z_{yy}$.

19. If $v = \frac{1}{\sqrt{t}}e^{-x^2/4a^2 t}$, Prove that $\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2}$.

20. If $\theta = t^n e^{-r^2/4t}$, what value of n will make $\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)\right] = \frac{\partial\theta}{\partial t}$.

21. If $x^x y^y z^z = c$, show that $z_{xy} = -\left[x\log_e(ex)\right]^{-1}$, when $x = y = z$.

22. If $z = \log_e\left(e^x + e^y\right)$, Show that $rt - s^2 = 0$ where $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$.

23. If $v = \log_e(x^2 + y^2 + z^2)$, prove that $(x^2 + y^2 + z^2)\left[v_{xx} + v_{yy} + v_{zz}\right] = 2$.

24. If $u = \log_e\left[x^3 + y^3 + z^3 - 3xyz\right]$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$.

25. If $w = r^m$, prove that $w_{xx} + w_{yy} + w_{zz} = m(m+1)r^{m-2}$ where $r^2 = x^2 + y^2 + z^2$.

26. Verify that v satisfies Laplace's equation $v_{xx} + v_{yy} + v_{zz} = 0$ if

a) $v = e^{3x+4y} \cos 5z$

b) $v = (x^2 + y^2 + z^2)^{-1/2}$

c) $v = \cos 3x \cos 4y \sinh 5z$

II. COMPOSITE FUNCTIONS

Total Derivative:

If $u = f(x, y)$ where $x = \phi(t)$ and $y = \psi(t)$ then $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$.

Examples

1. Find the differential of the following functions:
   a) $f(x, y) = x\cos y - y\cos x$
   b) $f(x, y) = e^{xyz}$

2. Find $\frac{du}{dt}$ for the following functions:
   a) $u = x^2 - y^2,\ x = e^t \cos t,\ y = e^t \sin t$ at $t = 0$.
   b) $u = x^y$, when $y = \tan^{-1} t$, $x = \sin t$
   c) $u = \sin(e^x + y)$, $x = f(t), y = g(t)$
   d) $u = \tan^{-1}(y/x),\ x = e^t - e^{-t},\ y = e^t + e^{-t}$
   e) $u = xy + yz + zx, x = 1/t,\ y = e^t$ and $z = e^{-t}$
   f) $u = x^3 y e^z$ where x = t, y = t^2 and $z = \log_e$ t, at t = 2.

3. Find $\frac{du}{dt}$ and verify the result by direct substitution.
   a) $u = \sin\left(\frac{x}{y}\right)$ and $x = e^t, y = t^2$.
   b) $u = x^2 + y^2 + z^2,\ x = e^{2t},\ y = e^{2t}\cos 3t$ and $z = e^{2t}\sin 3t$

4. If $u = \sin^{-1}(x - y),\ x = 3t,\ y = 4t^3$, show that $\frac{du}{dt} = 3(1 - t^2)^{-\frac{1}{2}}$. Also verify the result by direct substitution.

5. The altitude of a right circular cone is $15cm$ and is increasing at $0.2 cm/s$. The radius of the base is $10cm$ and is decreasing at $0.3 cm/s$. How fast is the volume changing?

6. Find the rate at which the area of a rectangle is increasing at a given instant when the sides of the rectangle are 4 ft and 3 ft and are increasing at the rate of 1.5 fts^{-1} and 0.5 fts^{-1} respectively.

7. In order that the function $u = 2xy - 3x^2 y$ remains constant, what should be the rate of change of y w.r.t. t, given x increases at the rate of 2cm/sec at the instant when x = 3cm and y = 1cm.

Partial differentiation of Composite functions:

If $u = f(x,y)$ where $x = g(s,t)$ and $y = h(s,t)$ then

$$\frac{\partial u}{ds} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial u}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

Examples

1. If $u = x^2 - y^2$, $x = 2r - 3s + 4$, $y = -r + 8s - 5$ find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$.
2. If $W = u^2 v$ and $u = e^{x^2 - y^2}$, $v = \sin(xy^2)$ find $\frac{\partial W}{\partial x}$ and $\frac{\partial W}{\partial y}$.
3. If $u = u\left[\frac{y-x}{xy}, \frac{z-x}{xz}\right]$, show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.
4. If $u = F(x-y, y-z, z-x)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.
5. Prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$ if z is a function of x and y and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$.
6. If $V = f(r,s,t)$ and $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$ show that $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = 0$.
7. If $x = u + v + w, y = vw + wu + uv, z = uvw$ and F is a function of x, y, z show that
$$u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\frac{\partial F}{\partial w} = x\frac{\partial F}{\partial x} + 2y\frac{\partial F}{\partial y} + 3z\frac{\partial F}{\partial z}$$
8. If z is a function of x and y and $x = e^u \cos v$, $y = e^u \sin v$.

   Prove that (i) $x\frac{\partial z}{\partial v} + y\frac{\partial z}{\partial u} = e^{2u}\frac{\partial z}{\partial y}$ $\quad (ii)\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right]$

III. JACOBIAN

If u and v are functions of two independent variable x and y then the Jacobian of u , v w.r.to x , y is denoted by

$$\frac{\partial(u,v)}{\partial(x,y)} \text{ or } J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Properties of Jacobians

1. If u,v are functions of x,y and x,y are functions of u,v
$J=\frac{\partial(u,v)}{\partial(x,y)}$ and $J'=\frac{\partial(x,y)}{\partial(u,v)}$ then $JJ'=1$.
2. If u,v are functions of r,s and r,s are functions of x,y then $\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial(u,v)}{\partial(r,s)}\times\frac{\partial(r,s)}{\partial(x,y)}$.

Examples

1. Find $\frac{\partial(u,v)}{\partial(x,y)}$ for the following:
   a) $u=e^x\sin y,\ v=x+\log_e(\sin y)$.
   b) $u=x+y,\ \ y=uv$
2. If $x=a\cosh\xi\cos\eta,\ y=a\sinh\xi\sin\eta$ Show that $\frac{\partial(x,y)}{\partial(\xi,\eta)}=\frac{1}{2}a^2(\cosh 2\xi-\cos 2\eta)$.
3. Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ for the following:
   a) $u=x^2,\ v=\sin y,\ w=e^{-3z}$.
   b) $u=x^2+y^2+z^2,\ v=xy+yz+zx,\ w=x+y+z$.
   c) $u=x+3y^2-z^3, v=4x^2yz,\ w=2z^2-xy$ at (1,-1,0).
   d) $u=\frac{2yz}{x},\ v=\frac{3zx}{y},\ w=\frac{4xy}{z}$
4. If $y_1=\frac{x_2x_3}{x_1},\ y_2=\frac{x_3x_1}{x_2},\ y_3=\frac{x_1x_2}{x_3}$, then show that $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}=4$.
5. Find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ if
   a) $x=\frac{u^2-v^2}{2}, y=uv,\ z=w$.
   b) $u=x+y+z,\ uv=y+z,\ uvw=z,$
6. Verify that $JJ'=1$
   a) $x=e^u\cos v,\ y=e^u\sin v$.
   b) $x=u(1-v),\ y=uv$.
   c) $u=x+\frac{y^2}{x}, v=\frac{y^2}{x}$.
   d) $x=u, y=u\tan v,\ z=w$
7. Find $\frac{\partial(u,v)}{\partial(r,\theta)}$ when,
   a) $u=x^2-2y^2, v=2x^2-y^2$ and $x=r\cos\theta,\ y=r\sin\theta$

b) $u = x^2 - y^2, v = 2xy$ and $x = r\cos\theta,\ y = r\sin\theta$

c) $u = 2xy, v = x^2 - y^2$ and $x = r\cos\theta,\ y = r\sin\theta$

d) $u = 2axy$ & $v = a\left(x^2 - y^2\right)$ and $x = r\cos\theta,\ y = r\sin\theta$

8. Find $\frac{\partial(X,Y)}{\partial(x,y)}$ where $X = u^2 v$, $Y = uv^2$ and $u = x^2 - y^2$, $v = yx$

9. If $x = \sqrt{vw},\ y = \sqrt{wu},\ z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi, v = r\sin\theta\sin\phi,\ w = r\cos\theta$, calculate $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$.

10. If $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $v = \sin^{-1}x + \sin^{-1}y$, show that u & v are functionally dependent and find the functional relationship.

11. If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$, show that u & v are functionally dependent and find the functional relationship.

IV. TAYLOR'S SERIES FOR FUNCTION OF TWO VARIABLES

$$f(x,y) = f(a,b) + \Delta f(a,b) + \frac{\Delta^2 f(a,b)}{2!} + \frac{\Delta^3 f(a,b)}{3!} + \cdots\cdots \text{ where } \Delta = (x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}.$$

Examples

Expand the following functions upto second degree terms.

1. $f(x,y) = x^2 + xy + y^2$ in powers of $(x-1)$ and $(y-2)$.
2. $f(x,y) = (1+x+y)^{-1}$ in powers of $(x-1)$ and $(y-1)$.
3. $f(x,y) = e^x \cos y$ in powers of $(x-1)$ and $\left(y - \pi/4\right)$.
4. $f(x,y) = x^y$ in powers of $(x-1)$ and $(y-1)$ and also find $(1.1)^{1.1}$.
5. $f(x,y) = \tan^{-1}\left(y/x\right)$ in powers of $(x-1)$ and $(y-1)$. Hence compute $f(1.1, 0.9)$.
6. $f(x,y) = x^3 + xy^2 + y^3$ in powers of $(x-1)$ and $(y-2)$.
7. $f(x,y) = \cot^{-1}(xy)$ in powers of $(x+0.5)$ and $(y-2)$ and hence compute $f(-0.4, 2.2)$.
8. $f(x,y) = x^2 y + 3y - 2$ in powers of $(x+1)$ and $(y-2)$.
9. $f(x,y) = \sin(xy)$ in powers of $(x-1)$ and $\left(y - \pi/2\right)$
10. $f(x,y) = xy^2 + \cos xy$ in powers of $(x-1)$ and $\left(y - \pi/2\right)$.
11. $f(x,y) = x^2 y + \sin y + e^x$ about $(1, \pi)$.

12. $f(x,y) = e^x \sin y$ about $(-1, \pi/4)$.

V. MACLAURIN'S SERIES FOR FUNCTION OF TWO VARIABLES

$$f(x,y) = f(0,0) + \Delta f(0,0) + \frac{\Delta^2 f(0,0)}{2!} + \frac{\Delta^3 f(0,0)}{3!} + \cdots\cdots \text{ where } \Delta = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Examples

1. $\cos x \cos y$ in powers of x and y up to second degree terms.
2. Expand $e^y \log_e(1+x)$ in powers of x and y up to third degree terms.
3. Expand $e^{ax} \sin by$ in the ascending powers of x and y up to third degree terms.
4. Find the Maclaurin's expansion of $e^x \log_e(1+y)$ up to third degree terms.
5. Expand $\log_e(1+x+y)$ in the neighbourhood of (0,0) up to second degree terms.

VI. MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

A function $z = f(x,y)$ is said to have a maximum or minimum at $x = a, y = b$ as $f(a+h, b+k) < f(a,b)$ or $f(a+h, b+k) > f(a,b)$ for all values of h and k.

Necessary conditions for $f(x,y)$ to have a maximum or a minimum at (a,b) are $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Working rule to find the maxima and minima of $z = f(x,y)$

1) Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ and equate them to zero, solve these as simultaneous equations in x and y.

   Let $(a,b),(c,d)$........... be the roots of the simultaneous equations.
2) Calculate the values of $r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$ for each of points.
3) If $rt - s^2 > 0$ and $r < 0$ at (a,b), $f(x,y)$ has a maximum value.
4) If $rt - s^2 > 0$ and $r > 0$ at (a,b), $f(x,y)$ has a minimum value.
5) If $rt - s^2 < 0$ at (a,b), (a,b) is a saddle point.
6) If $rt - s^2 = 0$ at (a,b) then the case is doubtful and needs further investigation.

Examples

1. Find the extreme values of the following functions:

a) $f(x,y)=x^3+3xy^2-15x^2-15y^2+72x$.

b) $f(x,y)=2\left(x^2-y^2\right)-x^4+y^4$.

c) $f(x,y)=xy+\frac{a^3}{x}+\frac{a^3}{y}$

d) $x^4+y^4-x^2-y^2+1$

e) $x^2+2y^2+3z^2-2xy-2yz-2$

f) $f(x,y)=x^3y^2(1-x-y)$

g) $f(x,y)=x^3+y^3-3y-12x+20$

h) $f(x,y)=\cos x\cos y\cos(x+y)$

i) $x^2+y^2+6x+12=0$

j) $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$

2. Find the shortest distance from origin to the plane $x-2y-2z=3$.
3. Find the points on the surface $z^2=xy+1$ nearest to the origin.
4. Find the shortest distance from origin to the surface $xyz^2=2$.
5. Sum of three numbers is a constant. Prove that their product is maximum when they are equal.
6. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum.
7. Examine the function $f(x,y)=\sin x+\sin y+\sin(x+y)$, $x,y\in(0,\pi)$ for extreme values.
8. In a plane triangle find the maximum value of $\cos A\cos B\cos C$ where A,B and C are the angles of the triangle.
9. Find the minimum value of $x^2+y^2+z^2$ given $ax+by+cz=p$
10. A flat circular plate is heated so that the temperature at any point (x,y) is $u(x,y)=x^2+2y^2-x$. Find the coldest point on the plate.
11. A rectangular box open at the top is to have a volume 108 cubic meters. Find its dimensions if its total surface area is minimum.
12. The temperature T at any point (x,y,z) in space is $T(x,y,z)=kxyz^2$ where k is a constant. Find the highest temperature on the surface of the sphere $x^2+y^2+z^2=a^2$.