BMS COLLEGE OF ENGINEERING, BENGALURU
DEPARTMENT OF MATHEMATICS

UNIT-1: DIFFERENTIAL CALCULUS - 1

Polar Curves: -

Angle between radius vector and tangent

1. If ϕ be the angle between radius vector and the tangent at any point of the curve $r = f(\theta)$ then prove that $tan(\phi) = r\frac{d\theta}{dr}$.

2. Find the angle between the radius vector and the tangent for the following polar curves.

a) $r = a(1+\cos\theta)$ **Ans**: $\pi/2 + \theta/2$.

b) $r^2 = a^2 \sin^2\theta$ **Ans**: $\phi = \theta$

c) $\frac{l}{r} = 1 + e\cos\theta$ **Ans**: $\phi = \tan^{-1}\left[\frac{1+e\cos\theta}{e\sin\theta}\right]$.

d) $r^m \cos m\theta = a^m$ **Ans**: $\pi/2 - m\theta$

3. Find the angle between the radius vector and the tangent for the following polar curves. And also find slope of the tangent at the given point.

e) $\frac{2a}{r} = 1 - \cos\theta$ at $\theta = 2\pi/3$ **Ans**: $\phi = \frac{2\pi}{3}, \tan\psi = \sqrt{3}$,.

f) $r\cos^2(\theta/2) = a^2$ at $\theta = 2\pi/3$ **Ans**: $\phi = \frac{\pi}{6}$.

g) $r^2\cos(2\theta) = a$ **Ans**: $\phi = \frac{\pi}{2} - 2\theta;\ \psi = \frac{\pi}{2} - \theta$

Angle between curves

Angle of intersection of two polar curves = angle of intersection of their tangents denoted by α

$$\alpha = |\phi_2 - \phi_1| \quad \text{or} \tan\alpha = \left|\frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1 \tan\phi_2}\right|$$

4. Find the angle of intersection of the following pair of curves:

a) $r = sin\theta + cos\theta, \quad r = 2\,sin\theta$ **Ans:** $\pi/4$

b) $r^2\sin 2\theta = 4$ and $r^2 = 16\sin 2\theta$ **Ans**: $\pi/3$.

c) $r = a$ and $r = 2a\cos\theta$. **Ans**: $\pi/3$.

d) $r = \frac{a}{\log\theta}$ and $r = a\log\theta$ **Ans**: $\tan^{-1}\left(\frac{2e}{1-e^2}\right)$.

e) $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$ **Ans**: $\tan^{-1}3$.

f) $r = a$ and $r = 2a\cos\theta$. **Ans**: $\pi/3$.

g) $r = 3\cos(\theta)$ and $r = 1+\cos(\theta)$ **Ans**: $\frac{\pi}{6}$.

5. Show that the following pair of curves intersect each other orthogonally.

a) $r = a(1+\cos\theta)$ and $r = b(1-\cos\theta)$.

b) $r = a\cos\theta$ and $r = a\sin\theta$.

c) $r = 4\sec^2(\theta/2)$ and $r = 9\operatorname{cosec}^2(\theta/2)$.

d) $r^n = a^n\cos(n\theta)$ and $r^n = b^n\sin(n\theta)$.

e) $r^2 \sin 2\theta = a^2$ and $r^2 \cos 2\theta = b^2$.

f) $r = ae^\theta$ and $re^\theta = b$.

g) $\frac{2a}{r} = 1 + \cos\theta$ and $\frac{2b}{r} = 1 - \cos\theta$.

h) $r = a\cos\theta$ and $r = a\sin\theta$.

i) $r = a\theta$ and $r = \frac{a}{\theta}$

Length of the perpendicular from pole to the tangent for the polar curve

1. If p denotes the length of the perpendicular from pole to the tangent of the curve $r = f(\theta)$, then prove that $p = r\sin\phi$ and hence deduce that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$
2. Find the length of the perpendicular from the pole to the tangent for the following curves
   a) $r = a(1-\cos\theta)$ at $\theta = \pi/2$ **Ans**: $a/\sqrt{2}$
   b) $r = a(1+\cos\theta)$ at $\theta = \pi/2$. **Ans** $a/\sqrt{2}$
   c) $r^2 = a^2\cos 2\theta$ at $\theta = \pi$. **Ans** a.
   d) $r^2 = a^2\sec 2\theta$ at $\theta = \pi/6$. **Ans** $a/\sqrt{2}$.
   e) $\frac{2a}{r} = (1-\cos\theta)$ at $\theta = \pi/2$ **Ans**: $\sqrt{2}(a)$
   f) $r = a\sec^2(\theta/2)$ at $\theta = \pi/3$ **Ans:** $2a/\sqrt{3}$

Pedal Equation

Relation between r and p, obtained after elimination of θ.

1. Find the pedal equation of the following curves:
   a) $r = ae^{\theta\cot\alpha}$. **Ans:** $p = r\sin\alpha$
   b) $r(1-\cos\theta) = 2a$ **Ans**: $p^2 = ar$
   c) $r^m\cos(m\theta) = a^m$ **Ans**: $pr^{m-1} = a^m$.
   d) $\frac{l}{r} = 1 + e\cos\theta$ **Ans**: $\frac{1}{p^2} = \frac{e^2 - 1}{l^2} + \frac{2}{lr}$.
   e) $r^m = a^m(\cos m\theta + \sin m\theta)$ **Ans**: $\phi = \frac{\pi}{4} + m\theta$.

Curvature and Radius of Curvature

Curvature = $\kappa = \frac{d\psi}{ds}$.

Radius of curvature = $\rho = \frac{1}{\kappa}$; $\kappa \neq 0$.

Radius of curvature (Cartesian form), $\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$, where $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$

If $y_1 \to \infty$ then $\rho = \dfrac{\left[\left(\dfrac{dx}{dy}\right)^2 + 1\right]^{3/2}}{\dfrac{d^2x}{dy^2}}$

Radius of curvature in polar coordinates: $\rho = \dfrac{\left(r^2 + r_1^2\right)^{3/2}}{r^2 + 2r_1^2 - rr_2}$; where $r_1 = \dfrac{dr}{d\theta}$ and $r_2 = \dfrac{d^2r}{d\theta^2}$

1. Find the radius of curvature for the following curves:
   a. at the point $\left(3a/2, 3a/2\right)$ of the Folium $x^3 + y^3 = 3axy$.
   b. $xy^2 = a^3 - x^3$ at (a, 0) on the curve.
   c. $a^2y^2 = a^3 - x^3$ at (a, 0) on the curve.
   d. $x^3 + xy^2 - 6y^2 = 0$ at (3, 3) on the curve.
   e. Catenary $y = c\cosh\left(x/c\right)$ at $(0, c)$.
   f. $(a, 0)$ on the curve $y^2 = \dfrac{a^2(a-x)}{x}$ is $a/2$.
   g. $\left(\dfrac{a}{4}, \dfrac{a}{4}\right)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $\dfrac{a}{\sqrt{2}}$.
   h. $x = \dfrac{\pi}{2}$ on the curve $y = 4\sin x - \sin(2x)$ is $\dfrac{5\sqrt{5}}{4}$.
   i. $r(1+\cos\theta) = a$ **Ans**: $\dfrac{(2r)^{3/2}}{\sqrt{a}}$.
   j. $r^n = a^n \cos n\theta$ **Ans**: $\dfrac{a^n}{(n+1)r^{n-1}}$.
   k. $r = a(1+\cos\theta)$ **Ans**: $\dfrac{2}{3}\sqrt{2ar}$.
   l. $r^n = a^n \sin n\theta$ **Ans**: $\dfrac{a^n}{(n+1)r^{n-1}}$.
   m. $r^2 \sec 2\theta = a^2$
2. Find the radius of curvature for $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at $(a, 0)$ and $(0, b)$.
3. Find the radius of curvature for $y^2 = 4ax$ at (x, y).
4. If ρ is the radius of curvature for $y = \dfrac{ax}{a+x}$ then prove that $\left(\dfrac{x}{y}\right)^2 + \left(\dfrac{y}{x}\right)^2 = \left(\dfrac{2\rho}{a}\right)^{2/3}$.
5. If ρ_1 and ρ_2 are the radii of curvature at the extremities of a chord through the pole for the polar curve $r = a(1+\cos\theta)$, prove that $\rho_1^2 + \rho_2^2 = \dfrac{16a^2}{9}$.
6. Show that for the curve $r(1-\cos\theta) = 2a$, ρ^2 varies as r^3.

Taylor's series and Maclaurin's series

Taylor's Series:- $f(x)=f(a)+\sum_{n=1}^{n=\infty}\frac{(x-a)^n f^n(a)}{n!}$.

Taking $a=0$ in **Taylor's series**, we get **Maclaurin's series**.

Problems:

1. Expand $f(x)=2x^3+7x^2+x-6$ in powers of $(x-2)$.

   Ans : $f(x)=40+53(x-2)+19(x-2)^2+2(x-2)^3$

2. Expand $f(x)=e^x$ about $x=3$ **Ans:** $f(x)=e^3\sum_{n=0}^{\infty}\frac{(x-3)^n}{n}$

3. Calculate the approximate value of $\cos 32^0$ using Taylor's series expansion

   Ans: $\cos 32^\circ = 0.8482$.

4. Expand $\sin x$ in powers of $\left(x-\frac{\pi}{2}\right)$ and hence find the value of $\sin 91^0$.

   Ans: $\sin x = 1-\left(x-\frac{\pi}{2}\right)^2\frac{1}{2}+\left(x-\frac{\pi}{2}\right)^4\frac{1}{4}-\left(x-\frac{\pi}{2}\right)^6\frac{1}{6}+\cdots;\quad \sin 91^\circ \approx 0.9998$.

5. Expand $\cos x$ in powers of $\left(x-\frac{\pi}{3}\right)$ and hence find the value of $\cos 61^0$.

   Ans: $\cos x=\frac{1}{2}-\left(x-\frac{\pi}{3}\right)\frac{\sqrt{3}}{2}-\left(x-\frac{\pi}{3}\right)^2\frac{1}{4}+\left(x-\frac{\pi}{2}\right)^3\frac{\sqrt{3}}{12}+\cdots;\quad \cos 61^0 \approx 0.4848$.

6. Expand $f(x)=\log_e x$, about $x=1$ and hence evaluate $\log_e 1.1$ correct to four decimal places.

   Ans: $f(x)=(x-1)-\frac{(x-1)^2}{2}+\frac{2}{3}(x-1)^3-\frac{3}{4}(x-1)^4+....;\ \log(1.1)\approx 0.0953$.

7. Prove that $\log_e \sin x = \log_e \sin a + (x-a)\cot a - \frac{1}{2}(x-a)^2 \operatorname{cosec}^2 a + ...$.

8. Expand $\tan^{-1} x$ in powers of $(x-1)$ up to four terms.

   Ans: $\tan^{-1} x = \frac{\pi}{4}+\frac{1}{2}\left\{(x-1)-\frac{(x-1)^2}{2}-\frac{(x-1)^3}{6}\right\}$

9. Expand $f(x)=\tan x$ about $x=\frac{\pi}{4}$ **Ans:** $f(x)=1+2\left(x-\frac{\pi}{4}\right)+2\left(x-\frac{\pi}{4}\right)^2+.......$

10. Expand $f(x)=\log_e \cos x$ about $x=\frac{\pi}{3}$ **Ans:** $f(x)=-\log_e 2-\sqrt{3}\left(x-\frac{\pi}{3}\right)-2\left(x-\frac{\pi}{3}\right)^2+.......$

Maclaurin's series:-

1. Obtain the Maclaurin's series expansion of the following functions:

   (i) e^x. (ii) $\sin x$. (iii) $\cos x$. (iv) $\sinh x$. (v) $\cosh x$ (vi) $\sin^{-1} x$ (vii) $\tan^{-1} x$

2. Determine the approximate value of π using the Maclaurin's series expansion of $\sin^{-1} x$.

3. Obtain the Macluarin's series of

   a. $f(x)=e^x \sin^2 x$ upto x^5. **Ans:** $f(x)=x^2+x^3+\frac{1}{6}x^4-\frac{1}{6}x^5+\cdots$.

b. $f(x) = \tan x$. **Ans:** $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + ...$

c. $f(x) = e^x \log_e(1+x)$. **Ans:** $y = x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{9x^5}{5} - \frac{35x^6}{6} + \cdots$.

d. $e^{\sin x}$ up to the term containing x^4. **Ans:** $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$.

4. Expand the following functions in ascending powers of x.

(i) $\log_e(1+x)$ and hence deduce that $\log_e\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + ...$.

(ii) $f(x) = \log_e \sec x$ upto x^6. **Ans:** $f(x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + ...$

(iii) $f(x) = a^x$. **Ans:** $a^x = 1 + x\log_e a + \frac{1}{2}(x\log_e a)^2 + \cdots + \frac{1}{n}(x\log_e a)^n + \cdots$.

(iv) $f(x) = e^{ax}\sin bx$ in ascending power of x. **Ans:** $e^{ax}\sin bx = bx + abx^2 + \frac{b(3a^2 - b^2)}{3}x^3 + ...$

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(v) $f(x) = \tan x$. **Ans:** $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + ...$

(vi) $f(x) = \frac{e^x}{1+e^x}$ upto x^3 **Ans:** $\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{1}{4}x - \frac{1}{8}\frac{x^3}{3!} + ...$

5. Prove that $\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + ...$

6. Using Macluarin's series expansion, show that

(i) $e^x\cos x = 1 + x - \frac{2x^3}{3} - 2^2\frac{x^4}{4} - 2^2\frac{x^5}{5} + 2^3\frac{x^7}{7} + ...$.

(ii) $\log_e(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + ...$

(iii) $\log_e(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + ...$