NAMRATHA. M.

MFCS

BOOK-1

09.01.2018

MFCS

1. Counting & Probability
2. Logics – Proposition logic, Predicate logic
3. Algebra – Linear algebra – Basis, dimension, matr
   Group, Ring, Fields
   Number Theory
4. Graph Theory
5. Complexity Theory – P & NP problems

BOOKS:

Discrete Mathematics – Kneth Rosan

" – C.L. Liu

Graphs – Narsingh Deo

COUNTING:

Two events

Product Rule $m, n \rightarrow m \times n$

Sum Rule $m, n \rightarrow m + n$

M. Tech 1 – 28 – 1 CR from 1

M. Tech 2 – 28 – 1 CR from 2

} 28 x 28 ways

1 CR from both – 28 + 28 ways

Permutation : ^nP_r (n, r)

Ordered arrangement of 'r' objects out of 'n' objects

a b c — 6 ways

a b c   b a c   c a b

a c b   b c a   c b a

$\boxed{}$ $\boxed{}$ $\boxed{}$ ---- $n-(r-1)$

n $n-1$ $n-2$

↓ ↓ ↓

$3 \times 2 \times 1 = \underline{\underline{6 \text{ ways}}}$

$$^nP_r = \frac{n!}{(n-r)!}$$

$$= \frac{n \times (n-1) \times (n-2) \text{ --- } (n-(r-1)) \times (n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

Distribution of 'r' distinct objects into 'n' distinct cells.

Each cell can hold max. 1 object (no repitition)

5 class rooms $\longrightarrow$ 4 classes $= {}^5P_4$

$5 \times 4 \times 3 \times 2 \times 1 = \underline{\underline{120}}$

Suppose, each cell can hold any number of objects. (instead of 1)

$= n^r$

(ordered collection of 'r' objects out of 'n' objects with repitition)

Ordered collection of 28 students $(n = 28, r = 28)$

$^{28}P_{28} = \underline{\underline{28!}}$

Putting 'n' objects in a line $- \underline{\underline{n!}}$

No of ways of putting 'n' objects in a ring

$= \frac{n!}{n} = \underline{\underline{(n-1)!}}$

All 'n' objects are treated as one single object

u v x ..   v x y

10.01.2018

1. Different 9 digit nos are possible

1, 2, 3, ---- , 8, 9

$^9P_9 = \underline{\underline{9!}}$ → without repitition

$\underline{\underline{9^9}}$ → with repitition

Atleast one digit should be repeated

$= \underline{\underline{9^9 - 9!}}$

<u>Lemma</u>: Let there be 'n' distinct objects q_1 objects of one kind, q_2 objects of 2nd kind, ------ q_t objects of t-th kind

Permutation of 'n' objects $= \dfrac{n!}{q_1! q_2! --- q_t!}$

<u>Proof</u>:

Required answer = x

q_1 objects of one kind is changed to q_1 distinct objects

Required permutation $= x \times q_1!$

Repeat the same for $q_2 = x \times q_2!$

Repeat ---- for all types of objects

$x \cdot (q_1! \, q_2! \, --- \, q_t!) = n!$

$$x = \frac{n!}{q_1! q_2! --- q_t!} \xrightarrow{} \text{answer}$$

$\dfrac{(k!)!}{(k!)^{(k-1)!}}$ is a $\dfrac{\text{number}}{\text{integer}}$ or $(k!)!$ is divisible $(k!)^{(k-1)!}$

Suppose $n = k!$, $\dfrac{n!}{(n!)^{(k-1)!}}$

$= k(k-1)!$

$k!$ objects – k objects of 1st type, k objects of 2nd type ---- k objects of $(k-1)!$-th type

$$= \frac{(k!)!}{k!k! ---- k! \, [(k-1)!] \text{ times}}$$

$\left.\begin{array}{l}\text{Numerator – number} \\ \text{Denominator – number}\end{array}\right\}$ since both are permutations

<u>COMBINATIONS</u>: Selection of 'r' objects out of 'n' distinct objects – no ordering

$$^nC_r = \frac{n!}{(n-r)! \, r!}$$

$${}^nP_r = \frac{n!}{(n-r)!}$$

Selecting 3 r out of 26 n alphabets

DAB BAD DBA ABD BDA ADB

$\underbrace{\text{DAB BAD DBA ABD BDA ADB}}_{}$

6 permutations – $3!$ ⇒ $r!$

↓

this will be replaced by 1 when order is not required

$$\therefore {}^nC_r = \frac{n!}{(n-r)!\,r!}$$

Distribution of 'r' non-distinct objects into 'n' distinct cells → $\underline{\underline{{}^nC_r}}$ $(r<n)$

Putting 2 x (same x x) into 4 distinct cells

1	2	3	4
x	x		
	x	x	

— only one since both objects are x $\boxed{\begin{matrix}1 & 2\\ 2 & 1\end{matrix}}$

⋮

$= {}^4C_2$

3 numbers are selected from 1, 2, 3 - - - 300 such that their sum is divisible by 3. How many ways?

1	3	6	9	mod 3 = 0	100 numbers
2	1	4	7	mod 3 = 1	100 — " —
3	2	5	8	mod 3 = 2	100 — " —

$\boxed{\text{mod } 3 = 3}$

$$100C3 + 100C3 + 100C3 + 100C_1 \times 100C_1 \times 100C_1$$

$$(100C_1)^3$$

4 1 number from category 1, 1 no from category 2 & 1 no from category 3

Other combinations of 1, 2 & 3 are not possible such as 2 nos from 2 & 1 no from 3

Number of different 32-bit sequence in which there is exactly seven 1's (rest are 0's). How many ways?

$\underline{\underline{32C7}}$

Distribution of 7 (non-distinct) 1's into 32 cells

12.01.2018

Selection of 'r' objects out of 'n' distinct objects with repitition $\underline{\underline{{}^{n+r-1}C_r}}$

Proof:

Distribution of 'r' non-distinct objects in ~~which~~ 'n' distinct cells in which each cell can hold any no of objects.

↳ Represent using bar (|) and star (*)

$n = 4$, $(n-1)$ bar (|), $r = 6$

(1) ** | | | ****

Interpretation: First object exists before first bar

Can also choose 'n' bars

1 | 2 | 3 | 4 | OR | 1 | 2 | 3 | 4

(2) * | * | * | ***

(3) ** | ** | ** |

(5) ****** | | |

(6) | | | ******

We have n-1 bars (|), r stars (*)

Ordered arrangement of 'n-1' bars & 'r' stars

$$= \frac{(n+r-1)!}{(n-1)!\, r!} \quad \text{(previous permutation formula)}$$

(1)

$$x_1 + x_2 + x_3 + x_4 = 20$$

$x_i \geq 0$ (integer) – only integer solution

Distribution of 20 1's into 4 variables (cells)

$$= {}^{20+4-1}C_{20} = \underline{\underline{{}^{23}C_{20}}}$$

$$= \frac{23!}{20!\, 3!} = \frac{23 \times \cancel{22}^{11} \times \cancel{21}^{7} \times \cancel{20!}}{\cancel{20!} \times \cancel{3} \times \cancel{2}} = \underline{\underline{23 \times 77}}$$

(2) 3 distinct dice are thrown – find the total no of possible outcomes?

Dice 1 : 1 2 3 4 5 6

Dice 2 : 1 2 3 4 5 6 $\Longrightarrow \underline{\underline{6^3}}$

Dice 3 : 1 2 3 4 5 6

(3) 3 indistinguishable dice $= {}^{3+6-1}C_3 = \underline{{}^{8}C_3}$

Distribution of 3 non-distinct (3 dice) into 6 cells (6 outcomes) in which each outcome can be repeated any no of times.

$n = 6,\ r = 3$ $\qquad$ ${}^{6+3-1}C_3 = \underline{\underline{{}^{8}C_3}}$

③ Distribute 7 apples & 5 oranges to 4 children

apples ↓ non-distinct $\qquad$ oranges ↓ non-distinct

$r = 7,\ n = 4$ $\qquad$ $r = 5,\ n = 4$

${}^{4+7-1}C_7$ $\qquad$ ${}^{5+4-1}C_5$

${}^{10}C_7$ $\qquad$ ${}^{8}C_5$

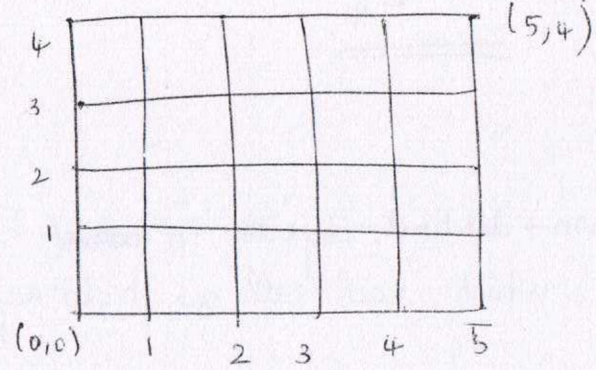
$\underline{{}^{10}C_7 \times {}^{8}C_5}$ (Product Rule)

Order of the objects within the cell is not considered here. (above examples)

Distribution of 'r' distinct objects into 'n' distinct cells where within the cell order of the objects matter.

$n \times (n+1) \times (n+2) - - - - - (n+r-1)$ ways

Distribution of 'r' non-distinct objects into 'n' non-distinct cells = One way (Trivial case)



Go from (0,0) to (5,4) – total no of paths

① Without any restriction on number of paths $= \infty$

② Can move only East or North | col (U) ↑ N, row (R) → E

No of R's is constant = 5

No of U's is constant = 4

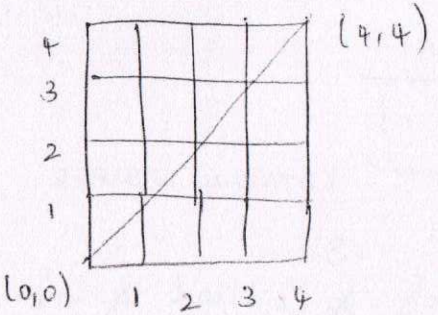
Path consists of total 9 steps

① 1 - 2 - 3 - 4 - 5 → ↑ 1 2 3 4

② 1 - 2 - 3 - 4 → ↑ 1 → 5 ↑ 2 3 4

Arrangement of 9 steps in which 5 steps are R are 4 steps are U = 9!

16.01.2018

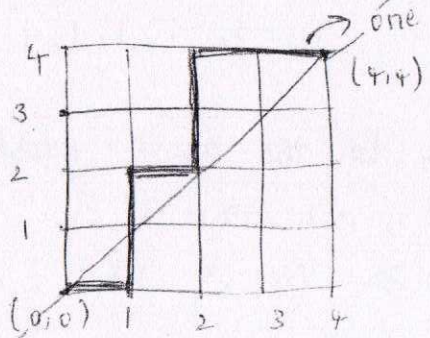


N, E

direction movement allowed $= \frac{(4+4)!}{4!\,4!}$

<u>No</u> of paths which crosses the line $y=x$ (diagnol) - atleast once

Ex:



Line $y=x$

no fixed length

Few Paths are:

① RUURUURR ② UUUURRRR

right up

③ RUUURRUUR

Difficult to find paths ∴ Map to some other paths for which we can find solution

One-to-one mapping $f: x \to y$ (onto function)

After crossing $y=x$ line, swap U & R

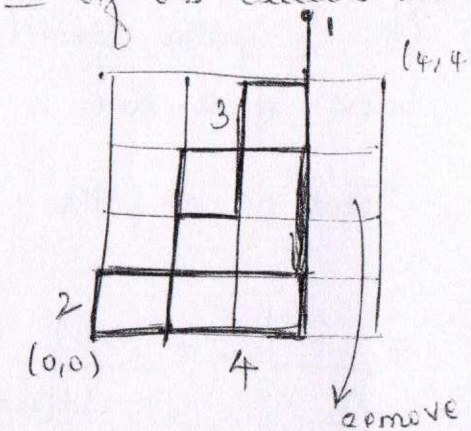
Path ① RUU | RUURR
swap U&R
RUU | URRUU

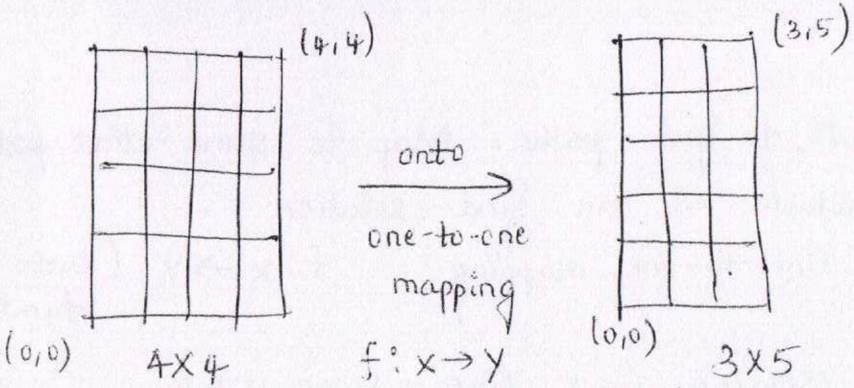
② U | UUU RRRR
swap
URRRUUUU

③ RUU | URRUR
swap
RUU RUURU

Swapping happens when <u>no</u> of U's exceeds the <u>no</u> of R's

④ $\underbrace{RRR}_{3}\underbrace{UUUU}_{4}$ | R
swap
RRRUUUUU





No of paths from (0,0) to (3,5) in 3X5 matrix

$$= \frac{8!}{3!\,5!}$$

In terms of 'n' $= \frac{2n!}{(n-1)!\,(n+1)!}$ → no of paths which crosses the line $y=x$

No of paths from (0,0) to (n,n) which lie below $y=x$ line CATALAN NUMBER

= Total no of paths − No of paths which crosses the line $y=x$

$$= \frac{2n!}{n!\,n!} - \frac{2n!}{(n-1)!\,(n+1)!} = \left[\frac{1}{(n)} - \frac{1}{n+1}\right]\frac{2n!}{n!\,(n-1)!}$$

$$= \underbrace{\frac{1}{(n+1)} \cdot \frac{2n!}{n!\,n!}}_{\text{CATALAN NUMBER}}$$

No of sequences of $\cancel{n}$ 3 1's and $\cancel{n}$ 3 −1's such that in any prefix no of −1's does not exceed the no of 1's

For example: $\left.\begin{array}{l} 1, -1, 1, -1, 1, -1 \\ 1, 1, 1, -1, -1, -1 \\ 1, 1, -1, -1, 1, -1 \end{array}\right\}$ ✓

1, −1, −1, 1, 1, −1 — X

Mapping this to the above problem,

$1 \to R$, $-1 \to U$

No of R's ≥ No of U's

$$= \frac{1}{(3+1)} \cdot \frac{(2\cdot 3)!}{3!\,3!} = \frac{1}{4} \cdot {}^6C_3$$

Matrix - Chain Multiplication - Dynamic Programming or Brute force Method

3 matrics $(A(BC))$ or $((AB)C)$ → 2 ways of paranthesis

4 matrices $(((AB)C)D)$

$((AB)(CD))$

(correlate to R's & U's)

No of ways of paranthesis

= No of '(' (R) = No of ')' (U) and

no of '(' $\geq$ no of ')' in any prefix

(order of closing is imp)

For 3 matrices $= \frac{1}{2+1} \, 4C_2 = \frac{1}{3} \cdot \frac{4!}{2!\,2!} = \frac{\cancel{4} \times 3 \times \cancel{2}}{\cancel{3} \times \cancel{2} \times 2}$

$= \underline{\underline{2 \text{ ways}}}$

For 'n' matrices $= \frac{1}{n} \frac{2(n-1)!}{(n-1)!\,(n-1)!}$

no of ways of paranthesizing (not multiplying) $= \frac{1}{n} \cdot {}^{2(n-1)}C_{(n-1)}$

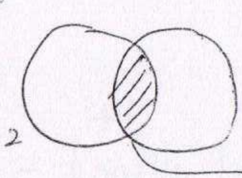
↓

Comparison { Brute force - Exponential time (coz nC_r)

DP algo $= n^3$

17·01·2018

Principles of Inclusion & Exclusion :



Events A_1 & A_2

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Generalization for 'n' events

$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i,j \leq n} |A_i \cap A_j|$

$A_1 \cap A_2$ and $A_2 \cap A_1$ are same - to be included only once

$+ \sum |A_i \cap A_j \cap A_k| \cancel{+} - - - (-1)^{n+1} |A_1 \cap A_2 \cap A_3 - - - A_n|$

Can be proved using mathematical induction

Find the no of integers which are divisible by 2, 3, 5, or 7 and less than 100 (≤ 100).

A_i = set of integers divisible by i

$|A_2 \cup A_3 \cup A_5 \cup A_7| = |A_2| + |A_3| + |A_5| + |A_7|$

$- (|A_2 \cap A_3| + - - -) \xrightarrow{} 6$ terms $4C2$

$+ (|A_2 \cap A_3 \cap A_5| + - - - -) = 4C3$

$|A_i| = \left\lfloor \frac{100}{i} \right\rfloor$ $\quad |A_i \cap A_j| = \left\lfloor \frac{100}{i \cdot j} \right\rfloor$

$|A_i \cap A_j \cap A_k| = \left\lfloor \frac{100}{i \cdot j \cdot k} \right\rfloor$

$$= \overset{50}{\left\lfloor \frac{100}{2} \right\rfloor} + \overset{33}{\left\lfloor \frac{100}{3} \right\rfloor} + \overset{20}{\left\lfloor \frac{100}{5} \right\rfloor} + \overset{14}{\left\lfloor \frac{100}{7} \right\rfloor} - \cdots$$

$$- \cdots - \underbrace{|A_2 \cap A_3 \cap A_5 \cap A_7|}_{=0}$$

No of primes ≤ 100

$\sqrt{100} = 10$

$2, 3, 5, 7 \rightarrow$ not divisible by any of these

$|A_2 \cup A_3 \cup A_5 \cup A_7| = x$

$|\overline{A_2 \cup A_3 \cup A_5 \cup A_7}| = 100 - x$

$= y$ (includes 1 because it is not divisible by 2, 3, 5, 7)

No of primes $= y - \underset{(1)}{1} + \underset{(2,3,5,7)}{4}$

Derangement: Each of a set of object has a position that it is forbidden to occupy and no 2 objects ha[ve] the same position. A derangement of these objects is a permutation of them such that no object is in it's forbidden position.

A_i = ith object is in it's forbidden position

No of arrangements in which atleast one object is it's forbidden position $= |A_1 \cup A_2 \cdots \cup A_n|$

Derangement $= \underbrace{n!}_{\text{total no sample space}} - |A_1 \cup A_2 \cdots \cup A_n| = |\overline{A_1 \cup A_2 \cdots \cup A_n}|$

$$|A_1 \cup A_2 \cdots A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| + \cdots (-1)^{n+1} |A_1 \cap A_2 \cdots \cap A_n|$$

$|A_i| = (n-1)!$ — only one object is in it's forbidden position, remaining objects can be placed anywhere

$|A_i \cap A_j| = (n-2)!$

$|A_i \cap A_j \cap A_k| = (n-3)!$

$$= \overset{\nearrow \underline{no}\text{ of terms} = n}{{}^nC_1 (n-1)!} - \overset{\nearrow \text{no of terms}}{{}^nC_2 (n-2)!} + {}^nC_3 (n-3)! -$$

$$- - - - (-1)^{n+1} \, {}^nC_n \cdot \underbrace{(n-n)!}_{0!}$$

$$\text{Derangement} = n! - \Big[{}^nC_1 (n-1)! - {}^nC_2 (n-2)! + - - - (-1)^{n+1} \, {}^nC_n \cdot (n-n)! \Big]$$

In how many ways can the letters $\alpha, \alpha, \alpha, \alpha, \beta, \beta, \beta, \gamma, \gamma$ be arranged so that all the letters of the same kind are not in the same block ?
(not together)

$\gamma\beta\alpha\alpha\alpha\alpha\beta\beta\gamma$ - not allowed
$\alpha\alpha\beta\beta\gamma\gamma\beta\alpha\alpha$ - not allowed

A_1 = set of arrangements in which all α's are in the same block

$A_2 = \beta$
$A_3 = \gamma$

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

Total arrangements $= \dfrac{9!}{4!\,3!\,2!}$
$(\alpha)\ (\beta)\ (\gamma)$

$|A_1| = \dfrac{6!}{3!\,2!\,1!}$ $\quad |A_2| = \dfrac{7!}{4!\,2!}$ $\quad |A_3| = \dfrac{8!}{4!\,3!}$

$|A_1 \cap A_2| = \dfrac{4!}{2!}$

$|A_2 \cap A_3| = \dfrac{5!}{4!}$

$|A_1 \cap A_3| = \dfrac{5!}{3!}$

$|A_1 \cap A_2 A_3| = 3!$ (3 objects)

19.01.2018

Ex: 11 scientists are working on secret project. They wis to lock up the documents in a cabinet so that cabinet can be opended if and only if 6 or more scientists are present. What is the smallest number of locks needed ? What is the smallest number of keys to the locks each scientist mus: have ?

Solution : 5 person — 1 lock

$\underline{No}$ of locks $= \underline{\underline{{}^{11}C_5}}$ Key of this lock is give

5 scientists together must not be able to open the lock - one more joins can open

5 persons together = 1 lock
↓
key to this lock is given to another 6 persons

$\boxed{10C5}$

or $\underbrace{11C5}_{\text{total}} - \underbrace{10C4}_{\text{choose remaining 4 person}}$

<u>No</u> of keys = $\underline{10C5}$
↓
leave that person (11-1=10)

(Group of 5 where the scientist is not part)

Practically, not possible to carry so many locks and keys

<u>Shamire</u> <u>Secret</u> <u>Sharing</u>:

Secret (Number Locking System - 2154) is distributed to 'n' persons such that 't' or more person can compute the secret. (2154)

Less than 't' persons cannot compute the secret.

<u>Theorem</u>:

Given 't' points $(x_1, y_1), \ldots (x_t, y_t)$ in the 2-D plane there is one and only one polynomial of degree t-1 such that $y_i = q(x_i)$.

$y = a + bx + cx^2 \longrightarrow$ polynomial
↓
Three unknowns - to solve 3 points are required

< 3 cannot solve

$y = \underset{\downarrow \text{secret}}{5} + a_1 x + a_2 x^2 + \ldots a_{t-1} x^{t-1}$

$(x_1, y_1) \longrightarrow D_1$

$(x_2, y_2) \longrightarrow D_2$

|

$(x_n, y_n) \longrightarrow D_n$

Polynomial of degree $t-1$ → leave x_1 for a_1 term

$$f(x) = a_1 \frac{(x-x_2) \cdots (x-x_t)}{(x_1-x_2) \cdots (x_1-x_t)}$$

$$+ a_2 \frac{(x-x_1)(x-x_3) \cdots (x-x_t)}{(x_2-x_1)(x_2-x_3) \cdots (x_2-x_t)}$$

$$+ \cdots + a_t \frac{(x-x_1) \cdots (x-x_{t-1})}{(x_t-x_1) \cdots (x_t-x_{t-1})}$$

Let $x = x_1$

$y_1 = f(x_1) = a_1$ (other terms become 0)

$y_2 = a_2$

$\vdots$

$y_t = a_t$

$$f(x) = y_1 \frac{(x-x_2) \cdots (x-x_t)}{(x_1-x_2) \cdots (x_1-x_t)} + y_2 \frac{(x-x_1)(x-x_3) \cdots (x-x_t)}{(x_2-x_1)(x_2-x_3) \cdots (x_2-x_t)}$$

$$+ \cdots + y_t \frac{(x-x_1) \cdots (x-x_{t-1})}{(x_t-x_1) \cdots (x_t-x_{t-1})}$$

Using 't' points we can compute $f(x)$

Secret $\underline{S = f(0)}$ = constant term

$$f(x) = y_1 l_1(x) + y_2 l_2(x) + \cdots y_t l_t(x)$$

where $l_i(x) = \frac{(x-x_1)(x-x_2) \cdots (x-x_{i-1}) \cdots (x-x_t)}{(x_i-x_1)(x_i-x_2) \cdots (x_i-x_{i-1}) \cdots (x_i-x_t)}$

$$f(0) = y_1 \frac{x_2 \cdots x_t}{(x_2-x_1) \cdots (x_t-x_1)} - \cdots + y_t \frac{x_1 x_2 \cdots x_{t-1}}{(x_1-x_t) \cdots (x_{t-1}-x_t)}$$

↓

Secret

No need to use Kramer's rule to solve 'n' equations.

23.01.2018

PROBABILITY

Measures the uncertainity.

Sample Size (Space) (SS) : Set of all possible outcomes

$$\text{Prob} = \frac{\text{No of outcomes}}{|SS|}$$

Two coins :

$SS = \{HH, HT, TH, TT\}$

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= \frac{n(E \cup F)}{|SS|}$$

Mutually Exclusive : E & F are mutually exclusive if $E \cap F = \phi$

If one occurs, other doesn't occur.

$$E \cap E^c = \phi$$

$$P(E \cup E^c) = 1 = P(E) + P(E^c) - \underbrace{P(E \cap E^c)}_{0}$$

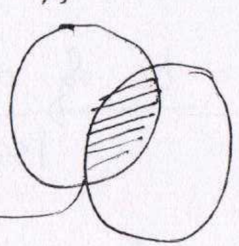
$$\boxed{P(E) + P(E^c) = 1}$$

Conditional Probability :

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

B → has already occured

$P(B)$ → total no of outcomes



$$P(A|B) = \frac{n(A \cap B)}{n(B)}$$

$$= \frac{n(A \cap B) / n(SS)}{n(B) / n(SS)} = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

$$P(B|A) \cdot P(A) = P(A|B) \cdot P(B) = P(A \cap B)$$

Independent Events :

$$P(A|B) = P(A)$$

↓

Occurence of A doesn't depend on B

Mutually exclusive events are not independent events. (Highly dependent)

(1) Husband watches the show – 80% – H
Wife " – 65% – W
Both " – 60% – $H \cap W$

Find the probability that wife watches the show if husband watches the show

$$P(W|H) = \frac{P(W \cap H)}{P(H)} = \frac{60\%}{80\%} = \frac{3}{4} = 75\%$$

Birthday Paradox:

How many person must be in a room before there is 50% chance that two of them are born on same day of the year?

A_i = event that i^{th} person's birthday is different from all j^{th} person $(j<i)$

$'N'$, ~~person~~ $= 365$

$'K'$ – total no of person

prob. that all $'K'$ person have different birthday

$$P(\underbrace{A_1 \cap A_2 \cap A_3 \cdots}_{A} \underbrace{A_K}_{B}) = P(A|B) \cdot P(B)$$

or

$$P(B|A) \cdot P(A)$$

$K-1$ terms

$$= P(A_K | A_1 \cap A_2 \cdots A_{K-1}) \cdot \underbrace{P(A_1 \cap A_2 \cdots A_{K-1})}_{\text{recursively apply conditional probability}}$$

$$= P(A_K | A_1 \cap A_2 \cdots A_{K-1}) \cdot P(A_{K-1} | A_1 \cap A_2 \cdots A_{K-2}) \cdot P(A_1 \cap A_2 \cdots A_{K-2})$$

$$= P(A_K | A_1 \cap A_2 \cdots A_{K-1}) \cdot P(A_{K-1} | A_1 \cap A_2 \cdots A_{K-2}) \cdot P(A_{K-2} | A_1 \cap A_2 \cdots A_{K-3}) \cdot P(A_1 \cap A_2 \cdots A_{K-3})$$

$$\vdots$$

$$= \underbrace{P(A_1)}_{=1} \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdots P(A_K | A_1 \cap A_2 \cap \cdots A_{K-1})$$

$$= \frac{N}{N} \cdot \left(\frac{N-1}{N}\right) \cdot \left(\frac{N-2}{N}\right) \cdots \left(\frac{N-(K-1)}{N}\right)$$

$$= \frac{N}{N} \cdot \left(1-\frac{1}{N}\right) \cdot \left(1-\frac{2}{N}\right) \cdots \left(1-\frac{K-1}{N}\right)$$

$$\approx \exp\left(-\frac{(1+2+\cdots K-1)}{N}\right) \qquad e^{-x} \approx 1-x \text{ for small}$$

$$= \exp\left(\frac{-(K(K-1))}{2N}\right)$$

Prob. that atleast 2 person out of K have same birthday $= 1 - \exp\left(\frac{-(K(K-1))}{2N}\right)$

$$1 - \exp\left(\frac{-(K(K-1))}{2N}\right) \geq \frac{1}{2} \to \text{already given}$$

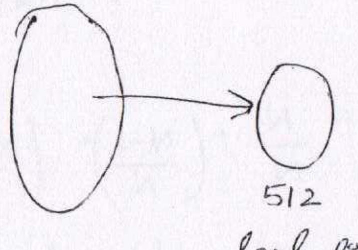
$$K = 1.17\sqrt{N} \approx 23$$

$N \to 365$

Application : Hash function in Cryptography

$H : \{0,1\}^* \xrightarrow{\text{SHA-512}} \{0,1\}^{512}$

any arbitrary string ($>>> 512$)



512

look like random

$\exists\ x_1, x_2$
$H(x_1) = H(x_2)$

It should be hard to find x_1 and x_2 such that $H(x_1) = H(x_2)$ — collision resistant property

Message Digest $= M = 2^{512}$

Worst case : $O(\underbrace{2^{512}}_{N})$ — collision occurs

$H(x)$: Birthday of x
$H(x_1)$: — " — x_1
$H(x_2)$: — " — x_2

In $\sqrt{N}$ trials — with probability $= 1/2$ or 50% one can find the collision

$O(\underbrace{2^{512}}_{N}) \implies O(\sqrt{2^{512}}) = O(2^{256})$

prob $= 1/2$

24·01·2018

Ex: Each of three men at a party throws their hat into the center of the room, hats are mixed up. Then each man randomly selects one hat. What is the probability that none of the three men selects his own hat?

Similar to derangement.

No. of ways none of 3 men selects his own hat

$= 3! - 3C_1 \cdot 2! + 3C_2 \cdot 1! - 3C_3 \cdot 0!$

$= 6 - 3\times 2 + 3 - 1 = \underline{\underline{2 \text{ ways}}}$

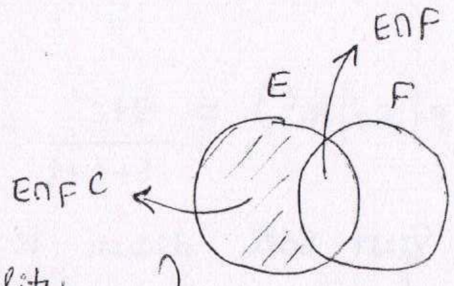
Probability $= \frac{2}{3!} = \frac{2}{6} = \frac{1}{3}$

OR

find $P(A \cup B \cup C)$ → 3 hats

Answer : $1 - P(A \cup B \cup C)$

Bayes Theorem :



E/F : Priori Probability (known)

F/E : Posterior Probability (to find)

} or vice versa

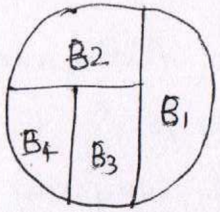
$$E = (E \cap F) \cup (E \cap F^c)$$

$$P(E) = P(E \cap F) + P(E \cap F^c) - P[\underbrace{(E \cap F) \cap (E \cap F^c)}_{0}]$$

$$P(E) = P(E|F) \cdot P(F) + P(E|F^c) \cdot P(F^c)$$

$$= \cancel{P(E|F)} [\cancel{P(F) + P(F^c)}]$$

$$P(F|E) = \frac{P(E \cap F)}{P(E)} = \frac{P(E|F) \cdot P(F)}{P(E|F) \cdot P(F) + P(E|F^c) \cdot P(F^c)}$$



B_1, B_2, B_3, B_4 - partitions of E

$\bigcup B_i = E$ & $B_i \cap B_j = \phi$

$$P(B_i|E) = \frac{P(B_i \cap E)}{\quad} = \frac{P(E|B_i) \cdot P(B_i)}{\quad}$$

$$= \frac{P(E|B_i) \cdot P(B_i)}{P(E|B_1) \cdot P(B_1) + P(E|B_2) \cdot P(B_2) + P(E|B_3) \cdot P(B_3) + P(E|B_4) \cdot P(B_4)}$$

$$E = (E \cap B_1) \cup (E \cap B_2) \cup (E \cap B_3) \cup (E \cap B_4)$$

* Probability model - Seldom Ross (Exercise problems) (FIRST 2 CHAPTERS)

Ex: ① A laboratory is 95% effective in detecting a certain disease when infact disease. However the test also yields a false positive result for 1% of the healthy person tested. If 0.5% of the population actually has the disease, what is the probability a person has the disease given that his test result is positive.

$D \rightarrow$ person has the disease

$P \rightarrow$ test result is positive

$$P(D|P) = \frac{P(P|D) \cdot P(D)}{P(P|D) \cdot P(D) + P(P|D^c) \cdot P(D^c)}$$

$$= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times 0.995}$$

(Urn)

2) Container contains 'b' ~~blue~~ black balls & 'r' red balls. One of the ball is drawn at random but while it is put back in the urn 'c' additional balls of the same color are put in with it. Now suppose that we draw another ball so that the probability that the first ball drawn was black given that second ball drawn is red.

$B_1 \rightarrow$ first ball drawn was black

$R_2 \rightarrow$ second ball drawn is red

$$P(B_1|R_2) = \frac{P(R_2|B_1)\cdot P(B_1)}{P(R_2|B_1)\cdot P(B_1) + P(R_2|B_1^c)\cdot P(B_1^c)}$$

$$P(B_1) = \frac{b}{b+r}$$

$$P(B_1^c) = 1 - \frac{b}{b+r} = \frac{r}{b+r}$$

$$P(R_2|B_1) = \frac{r}{b+c+r}$$

First ball drawn is black, while putting back - c additional black balls are put in.

$$P(R_2|B_1^c) = \frac{r+c}{b+c+r}$$

If first ball drawn is red, 'c' additional red balls are put back.

$$P(B_1|R_2) = \frac{\frac{r}{b+c+r}\cdot\frac{b}{b+r}}{\frac{r}{b+c+r}\cdot\frac{b}{b+r} + \frac{r+c}{b+c+r}\cdot\frac{r}{b+r}}$$

$$= \frac{\frac{br}{(b+r)(b+c+r)}}{\frac{br}{(b+c+r)(b+r)} + \frac{r(r+c)}{(b+c+r)(b+r)}}$$

$$= \frac{br}{br+r^2+rc} = \frac{br}{r(b+r+c)} = \frac{b}{b+r+c}$$

30.01.2018

ELT : transmit the signal. The Altigauge (A) manufacturing company makes 80% of ELT & B company makes 15% & C company makes 5%. The ELT made by A has 4% defects, B has 6% defects & C has 9% defects. TO a randomly selected ELT is tested and is found to be defective, Find the probability that it is made by A.

A : Selected ELT is found to be defective

B_1 : ELT is made by A company

$$P(B_1|A) = \frac{P(A|B_1)\cdot P(B_1)}{P(A|B_1)\cdot P(B_1) + P(A|B_2)\cdot P(B_2) + P(A|B_3)\cdot P(B_3)}$$

$$= \frac{0.04 \times 0.8}{0.04 \times 0.8 + 0.06 \times 0.15 + 0.09 \times 0.05}$$

RANDOM VARIABLES :

R.V is a function defined on sample space S.

$X : S \to I \longrightarrow$ Integer (part of Discrete random variable (RV))

X – Random Variable; S – domain; I – integer (range)

$X : S \to R \longrightarrow$ Real R.V

R – real no (part of continuous)

Ex: Two coins are tossed. $S = \{HH, HT, TH, TT\}$

Let X be a RV which denotes no of heads.

$$X : S \to \{0, 1, 2\}$$

$P[X=0] = 1/4$ (Only TT occurs)

$= p(0)$ no of heads = 0

$\downarrow$

Probability Mass function (PMF) $\boxed{\sum_i p(i) = 1}$

Discrete – PMF (Mass)

Continuous – PDF (Density) Density = Mass/Area

Cummulative Distribution Function (CDF)

$$F(a) = \sum_{i \le a} p(i)$$

Bernoulli's Distribution : (Part of discrete random variable)

$X : S \to C$

(countable set)

I is part of C $[I \subset C]$

Two possible outcomes: Success (S) (p) $S=1$; Failure (F) $(1-p)$ $F=0$

$$X : S \to \{\overset{F}{0}, \overset{S}{1}\}$$

$$P[X=1] = p = p(1)$$
$$P[X=0] = (1-p) = p(0)$$

$E[X]$ = expected value of RV X

$$= \frac{1 \cdot p(1) + 0 \cdot p(0)}{p + (1-p)} = p(1) = p$$

$$\boxed{\therefore E[X] = p}$$ (same as probability of success)

Variance (measure of spreadness)

$$= E[X - E[X]]^2$$

(squaring done in order to make sure that +ve & -ve terms don't cancel out)

To compute Variance:

1. Find expected value
2. Subtract from each value
3. Square all values
4. Sum up the values

$$X : S \to \{x_1, x_2, x_3\}$$

$$\text{Variance} = p(x_1)(x_1 - E[X])^2 + p(x_2)(x_2 - E[X])^2 + p(x_3) \cdot (x_3 - E[X])^2$$

$$\text{Variance} = E[X - E[X]]^2$$
$$= E\left[(X)^2 + (E[X])^2 - 2 \cdot X \cdot E[X]\right]$$
$$= E[X^2] + E[(E[X])^2] - E[2X \underbrace{E[X]}_{\text{constant}}]$$

① $E[cX] = cE[X]$ (c → constant)

② $E[c] = c$

using this, using this

$$= E[X^2] + (E[X])^2 - 2E[X]E[X]$$
$$= E[X^2] + (E[X])^2 - 2(E[X])^2$$
$$= E[X^2] - (E[X])^2$$
$$= \sigma^2$$

Standard Deviation (SD) $\sigma = \sqrt{\text{Variance}}$

Variance for Bernoulli's distribution:

$E[x] = p$

$E[x^2] = \underline{x_1^2 p(x_1) + x_2^2 p(x_2)}$

$= \cancel{p}\ 1^2 \cdot \overset{1}{p} + 0^2 \cdot (1-p)$

$= p$

Variance $= E[x^2] - (E[x])^2$

$= p - p^2 = \underline{\underline{p(1-p)}}$

31.01.2018

Geometric RV:

Ex: Probability that 10th time - first head

$\underbrace{\left(\frac{1}{2}\right)^9}_{\text{9 times we get Tail}} \cdot \frac{1}{2}$

All trials here are independent of each other

Independent trials and each trial has 2 possibilities p (success) and $(1-p)$ (failure)

X: (RV)

Number of trials required until first success comes

$X: S \rightarrow \{1, 2, ---\infty\}$

we may not get success

$P[X=n] = p(n) = (1-p)^{n-1} \cdot p$

n-1 failures to get success in nth trial

$E[x] = 1 \cdot p(1) + 2 \cdot p(2) + ---i \cdot p(i) + ----$

$= \sum_{i=1}^{\infty} i \cdot p(i)$

$= \sum_{i=1}^{\infty} \underbrace{i(1-p)^{i-1}} \cdot p$

$= -p \sum_{i=1}^{\infty} \frac{d}{dp}(1-p)^i$

swap

$= -p \cdot \frac{d}{dp} \sum_{i=1}^{\infty} (1-p)^i$ GP (till ∞) $= \frac{a}{1-r}$ $\quad \frac{a(r^n-1)}{r-1}$

$= -p \cdot \frac{d}{dp} \frac{1-p}{1-(1-p)} = -p \frac{d}{dp}\left(\frac{1}{p} - 1\right) = \frac{1}{p}$

Binomial Distribution :

'n' independent trials (already given) with 2 possibilities p & $(1-p)$

Ex: Prob. of getting 5 heads in 10 trials.

$$10C6 \cdot \underbrace{\left(\frac{1}{2}\right)^6}_{\text{success}} \cdot \underbrace{\left(\frac{1}{2}\right)^4}_{\text{failure}}$$

X: Denotes the number of successes in n-trials

$$X : S \rightarrow \{0, 1, 2, ---- n\}$$

In n-trials we may not get any success at all.

$$P[X=r] = p(r) = nC_r \cdot p^r (1-p)^{n-r}$$

$$E[X] = 0 \cdot p(0) + 1 \cdot p(1) + ---- n \cdot p(n)$$

$$= \sum_{i=0}^{n} i \cdot p(i)$$

$$= \sum_{i=0}^{n} i \cdot nC_i \cdot p^i (1-p)^{n-i}$$ (When $i=0$, term becomes 0)

$$= \sum_{i=1}^{n} i \cdot nC_i \cdot p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i \cdot \frac{n(n-1)!}{(n-i)!\,i!} \cdot p \cdot p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \frac{i \cdot (n-1)!}{(n-i)!\,i!} \cdot p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \frac{(n-1)!}{\underbrace{(n-i)!}_{n-1-(i-1)}\,(i-1)!} \cdot p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} (n-1)C_{(i-1)} \cdot p^{i-1} (1-p)^{\overset{n-1-(i-1)}{n-i}}$$

$$= np \sum_{j=0}^{n-1} (n-1)C_j \cdot p^j (1-p)^{n-1-j}$$ (Here $i-1=j$)

Rewrite in this form of binomial distribution from 0 to n-1

$$= np \cdot (p + (1-p))^{n-1}$$

$$= np \cdot (1)^{n-1}$$

$$= \underline{\underline{np}}$$

Ex: An aeroplane engine will fail when in flight with probability $(1-p)$. Suppose that aeroplane will make successful flight if atleast 50% of its engines remain operative. For what value of p is a 4-engine plane is preferable to 2-engine plane.

Probability that 4-engine plane will have a successful flight.

$P[x \geq 2] = 4C_2 \cdot p^2 \cdot (1-p)^2 + 4C_3 \cdot p^3 (1-p)^1 + 4C_4 \cdot p^4 \cdot (1-p)^0$
50% of 4

Probability that 2-engine plane will be successful

$P[x \geq 1] = 2C_1 \cdot p^1 (1-p)^1 + 2C_2 \cdot p^2 (1-p)^0$
50% of 2

$4C_2 \cdot p^2 \cdot (1-p)^2 + 4C_3 \cdot p^3 (1-p)^1 + 4C_4 \cdot p^4 \cdot (1-p)^0$

$> 2C_1 \cdot p^1 (1-p) + 2C_2 \cdot p^2 (1-p)^0$

$6p^2(1+p^2-2p) + 4p^3(1-p) + p^4 > 2p(1-p) + p^2$

$6p^2 + 6p^4 - 12p^3 + 4p^3 - 4p^4 + p^4 > 2p - 2p^2 + p^2$

$3p^4 - 8p^3 + 6p^2 > -p^2 + 2p$

$3p^4 - 8p^3 + \cancel{7}p^2 > 2p$

$3p^{\cancel{4}\,3} - 8p^2 + \cancel{7}p > 2$

Negative Binomial Distribution :

Toss a coin until 06 heads occur. What is the probability that it will happen in 15th trial?

$$14C_5 \cdot \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^9 \cdot \frac{1}{2}$$

$$n-1C_{x-1} \cdot p^{x-1} (1-p)^{n-x} \cdot p$$

$\downarrow$

x – no of successes

First find $x-1$ successes in $n-1$ trials & 1 success in nth trial.

$\left.\begin{array}{l}(\text{no of success})\ x \rightarrow \text{constant} \\ \text{no of trials} \rightarrow \text{to find}\end{array}\right\}$ similar to geometric RV

But in binomial, no of trials $\rightarrow$ fixed
To find $\rightarrow$ no of successes

06-02-2018

Negative Binomial is reverse of binomial

Binomial : No of successes given 'n' trials

Negative Binomial : No of trials required for

$$P[X=n] = {}^{n-1}C_{r-1}\, p^{r-1} \cdot (1-p)^{n-1-(r-1)} \cdot p$$

No. of failures required for r-success.

$$P[X=s] = {}^{r+s-1}C_{r-1}\, p^{r-1} \cdot (1-p)^{(r+s-1)-(r-1)} \cdot p$$

r → no of success
s → no of failures

Total : $\underline{\underline{r+s}}$ trails

$$\boxed{E[X] = \frac{r}{p}}$$

Hyper Geometry :
Consider a population of N individuals out of which K are male and N-K are female. A sample of n-individuals are randomly selected from this sample N. What is the probability that x males were selected ?

$\cancel{{}^{n}C_{K} \cdot {}^{N-n}C_{n-K}}$

K male - x males are selected

$$\left[{}^{K}C_{x} \cdot {}^{N-K}C_{n-x}\right] / {}^{N}C_{n}$$

remaining female

total no of samples

X

HG RV : denotes no of males from the sample of n

$$P[X=x] = \frac{{}^{K}C_{x} \cdot {}^{N-K}C_{n-x}}{{}^{N}C_{n}}$$

$$\boxed{E[X] = \frac{Kn}{N}}$$

$$\text{Variance} = \frac{Kn(N-n)(N-K)}{N^2(n-1)}$$

Binomial Distribution :

$$P[X=r] = {}^{n}C_{r} \cdot p^{r}(1-p)^{n-r}$$

Average $= np = \lambda$

As $n \to \infty$ and $r \to 0$ (very small)

$$P[X=r] = \frac{n!}{(n-r)!\,r!} \left(\frac{\lambda}{n}\right)^{r} \frac{(1-\lambda/n)^{n}}{(1-\lambda/n)^{r}}$$

$\to 1/e^{\lambda} = e^{-\lambda}$

≈ 1 since n is large

$$= \frac{n!}{(n-r)!\,r!} \cdot \frac{\lambda^{r}}{n^{r}} \cdot \frac{e^{-\lambda}}{1}$$

$$= \frac{n!}{(n-r)!\,n^{r}} \left(\frac{\lambda^{r} e^{-\lambda}}{r!}\right)$$

$$= \frac{n(n-1)\cdots\cancel{(n-r)!}}{\cdots} \left(\lambda^{r} e^{-\lambda}\right)$$

$$= \left[\frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-2}{n} \cdots \frac{n-(r-1)}{n}\right]\cdot\frac{e^{-\lambda}\lambda^{r}}{r!}$$

$$= \left[\frac{n}{n}\cdot\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots 1-\frac{(r-1)}{n}\right]\cdot\frac{e^{-\lambda}\lambda^{r}}{r!}$$

each term ≈ 1 since 'n' is very large

$$= \frac{e^{-\lambda}\lambda^{r}}{r!}$$

$\downarrow$ Poisson Distribution

<u>Poisson</u> <u>RV</u>: Denotes <u>no</u> of successes

$$P[X=r] = \frac{e^{-\lambda}\lambda^{r}}{r!}$$

All trials are random and independent.

$n \to \infty$ and $p \to 0$ $\to$ <u>no</u> of successes is very low.

$\downarrow$

<u>no</u> of trials is very large

Arrival rate in a network - poisson

<u>No</u> of packets - very large ($N \to \infty$)

Packets arrive in a particular network with probability $p \to 0$ (very low since there are several networks)

Probability that 'r' packets arrive in a network (NITT) $= \dfrac{e^{-\lambda}\cdot\lambda^{r}}{r!}$

Arrival of cars in Toll booth.

N = <u>no</u> of cars - very large

$p \to 0$ - because car arriving at a particular toll booth is very low (since there are many toll booths)

$$E[X] = \sum_{r=0}^{\infty}\frac{r\cdot e^{-\lambda}\lambda^{r}}{r!}$$

$(r \cdot p_r)$ — r: 0 to ∞, p_r: probabili

$$= \lambda\sum_{r=0}^{\infty}\frac{r\cdot e^{-\lambda}\lambda^{r-1}}{r!}$$

$$= \lambda e^{-\lambda}\sum_{r=0}^{\infty}\frac{r\cdot\lambda^{r-1}}{r!}$$

$$= \lambda e^{-\lambda}\sum_{r=1}^{\infty}\frac{r\cdot\lambda^{r-1}}{r!}$$

[$r=0$ the whole term becomes 0]

$$= \lambda e^{-\lambda}\sum_{r=1}^{\infty}\frac{\cancel{r}\cdot\lambda^{r-1}}{\cancel{r}(r-1)!}$$

$$= \lambda e^{-\lambda}\sum^{\infty} \lambda^{r-1}/$$

Let $r-1 = j$

$$= \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

$$= \lambda e^{-\lambda} \left[\frac{\lambda}{1!} + \frac{\lambda}{2!} + \cdots \frac{\lambda}{\infty!} \right]$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda}$$

$$= \underline{\underline{\lambda}}$$

→ expected value

07-02-2018

Cars arrive at Toll Booth at average rate of 150 per hour. (a) Find the probability that in a given minute no cars arrive.

(b) Probability that in a given minute 3 cars arrive.

Arrival of cars in a minute

Average rate $= \frac{150}{60}$ per minute $= \lambda$

(a) $P[x=0] = \lambda^0 e^{-\lambda} = e^{-\lambda}$

(b) $P[x=3] = \frac{\lambda^3}{3!} \cdot e^{-\lambda} = \frac{\lambda^3}{6} \cdot e^{-\lambda}$

(c) Probability that atleast 2 cars arrive

$$= 1 - P[x=0] - P[x=1]$$

$$= 1 - e^{-\lambda} - \lambda \cdot e^{-\lambda}$$

$$= \underline{\underline{1 - e^{-\lambda}(1+\lambda)}}$$

Reception desk receives phone calls at the rate of 4 calls per 10 minutes. Assume that calls are random and independent with constant average rate. What is the probability of receiving no calls in a 5 minute period?

$\lambda = 4$ calls / 10 minutes

$\lambda = 2$ calls / 5 minutes

$$P[x=0] = \frac{e^{-\lambda} \cdot \lambda^0}{0!} = e^{-\lambda} = \underline{\underline{e^{-2}}}$$

CONTINUOUS RANDOM VARIABLE :

$X : S \rightarrow R$ (continuous)

$P[X=i] = p(i) =$ point probability $= 0$

$\hookrightarrow$ infinite points

$\therefore$ We can't use $p(i)$ here as in discrete RV.

$f(x) =$ Probability Density Function (PDF)

$P[X=i] = 0 = p(i)$

$$P[a \leq X \leq b] = \int_a^b f(x)\,dx$$

Point probability :

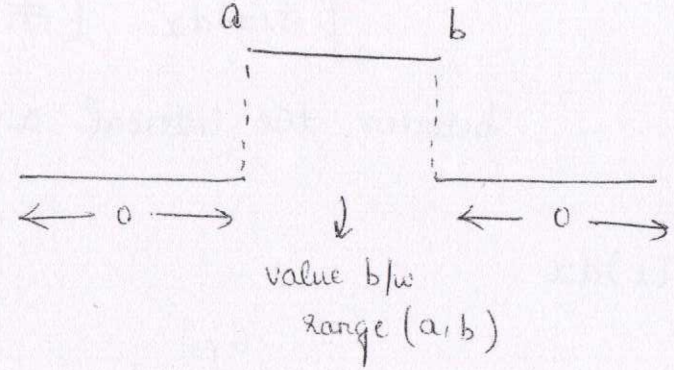
$$P[a \leq X \leq a] = \int_a^a f(x)\,dx = 0$$

Cumulative Distribution Function :

$$= F[a] = \int_{-\infty}^{a} f(x)\,dx$$

$$\underbrace{\frac{d}{da} F[a]}_{\text{CDF}} = \underbrace{f(a)}_{\text{PDF}}$$

1. UNIFORM RANDOM VARIABLE :



R.V X is uniformly distributed $a \leq x \leq b$

$$f(x) = \begin{cases} 0 & \cancel{x} \leq a \\ \frac{1}{b-a} & a \leq x \leq b \\ 0 & x \geq b \end{cases}$$

Total prob $= \int_{-\infty}^{\infty} f(x)\,dx$

$= \int_a^b 1/(b-a)\,dx$

$= \frac{1}{\cancel{b-a}} \cancel{(b-a)}$

$= \underline{\underline{1}}$

Expected Value $E[x] = \dfrac{\int_{-\infty}^{\infty} x f(x)\,dx}{\int_{-\infty}^{\infty} f(x)\,dx}$

$$= \frac{\int_{-\infty}^{\infty} x f(x)\,dx}{1} = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int^{a} x\, f(x)\,dx + \int_{b} x f(x)\,dx + \int^{\infty} x f(x)\,dx$$

Expected value $E[x] = \dfrac{\int_a^c x f(x)\,dx}{\int_a^c f(x)\,dx}$ $(\neq 1)$

between the interval a to c

$= \int_a^b x f(x)\,dx$

$= \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x\,dx$

$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$

$= \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) = \frac{(b-a)(b+a)}{(b-a)\cdot 2}$

$\therefore E[x] = \underline{\underline{\frac{a+b}{2}}}$

Variance $= E[\underbrace{X-\mu}_{\text{expected value}}]^2$

$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

$= \int^b (x-\mu)^2 \cdot \frac{1}{b-a}\,dx$

$= \frac{1}{b-a}\int_a^b (x^2+\mu^2-2\mu\cdot x)\,dx$

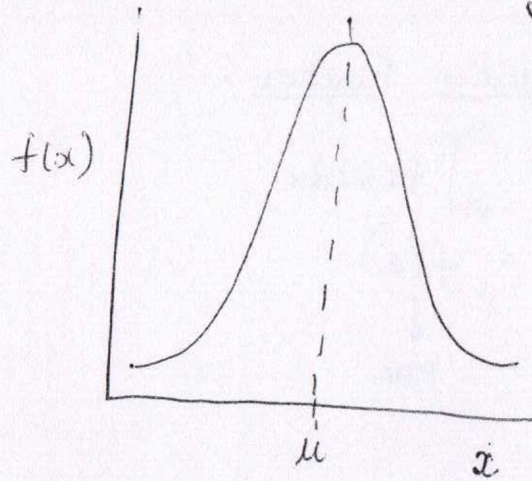
$= \frac{1}{b-a}\int_a^b \left[x^2+\left(\frac{b+a}{2}\right)^2 - 2\left(\frac{b+a}{2}\right)\cdot x\right] dx$

$= \frac{1}{b-a}\left\{\left[\frac{x^3}{3}\right]_a^b + \left(\frac{b+a}{2}\right)^2 - 2\left(\frac{b+a}{2}\right)\left(\frac{x^2}{2}\right)\right.$

$= \frac{1}{b-a}\left[\left(\frac{b^3-a^3}{3}\right) + \left(\frac{b+a}{2}\right)^2 - \cancel{2}\left(\frac{b+a}{\cancel{2}}\right)\left(\frac{b^2-a^2}{2}\right)\right]$

$= \underline{\underline{\frac{(b-a)^2}{12}}}$

2. <u>NORMAL</u> <u>DISTRIBUTION</u> : (Gaussian Distribution)
widely used



Symmetric about the line $x = \mu$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

mean

σ → SD

Ex: Marks distribution of students is normally distributed

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

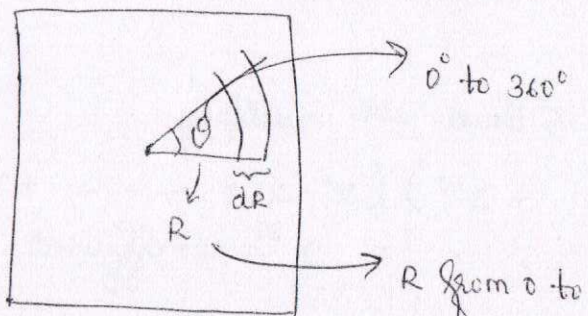
Let $\frac{x-\mu}{\sqrt{2}\cdot\sigma} = t$ $\therefore \frac{dx}{\sqrt{2}\cdot\sigma} = dt$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-t^2}\,dt = \frac{1}{\cancel{\sqrt{\pi}}} \underbrace{\int_{-\infty}^{\infty} e^{-t^2}\,dt}_{= I} \quad (1)$$

$$= \frac{1}{\cancel{\sqrt{\pi}}} \underbrace{\int_{-\infty}^{\infty} e^{-s^2}\,ds}_{} \quad (2) = I$$

$$(1) \times (2) \quad I^2 = \frac{1}{\cancel{\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(s^2+t^2)}\,ds\,dt$$

To integrate whole space



Polar co-ordinates

08.02.2018

Assignment Sample Solutions

11, 19

19.(d) $2 \le x_i \le 10$ $x_1 + x_2 + x_3 + x_4 + x_5 = 21$

$y_1 + y_2 + y_3 + y_4 + y_5 = 11$ (Give 2 to each)

$0 \le y_i \le 8$

All solutions, $r = 11$, $n = 5$

$^{5+11-1}C_r = \underline{\underline{^{15}C_{11}}}$

$(^5P_2)$

$y_i = 9$, $y_i = 10$, $y_i = 11$

$^5C_1 \times {^4C_2} + {^5C_1} \times {^4C_1}$ — 10 and 1 $\underline{\underline{^5C_1 \times {^4C_1}}}$ = 20 ways — Either y_1, y_2, y_3, y_4 or y_5 can be 11 − 5 $\underline{\underline{\text{ways}}}$

All solutions − ($y_i = 9$ + $y_i = 10$ + $y_i = 11$)

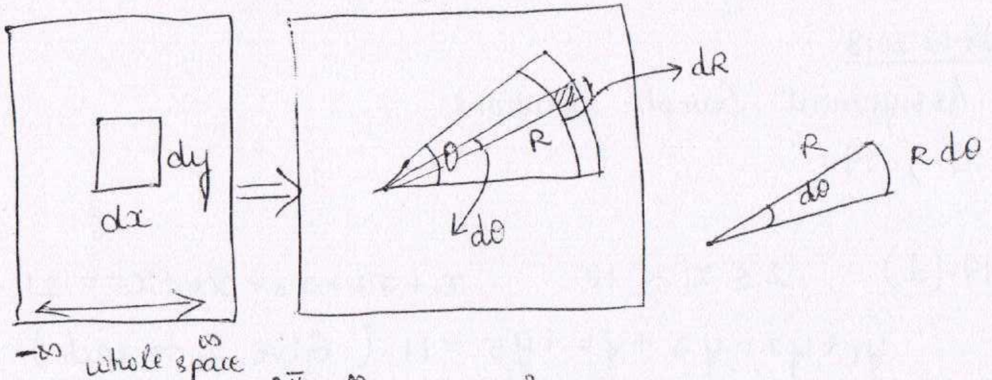
5 times we multiply

$(x^2+x^3+ - - - - + x^{10})(x^2+x^3+ - - - - + x^{10})$

$\downarrow$ $- - - - - -$ $= x^{21} =$ co-efficient

OR

Co-efficient of x^{21} in $(x^2+x^3+ - - - x^{10})^5$



$$I^2 = \cancel{\frac{1}{\pi}} \int_0^{2\pi}\int_0^{\infty} R e^{-R^2}\, d\theta\, dR$$

$R^2 = u$ $\qquad R\,dR = du/2$

$$= \cancel{\frac{1}{2\pi}} \int_0^{2\pi}\int_0^{\infty} e^{-u}\, du\, d\theta$$

$$= \frac{1 \cdot 1}{\cancel{2}\pi\, 2} \int_0^{2\pi} \left(-e^{-u}\right)_0^{\infty} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1\, d\theta = \frac{1}{2}\left[\theta\right]_0^{2\pi} = \frac{1}{\cancel{2}}(\cancel{2}\pi - 0)$$

$\therefore I^2 = \pi$

$I = \sqrt{\pi}$

$$I = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = \underline{\underline{1}}$$

$$\therefore \underline{\int_{-\infty}^{\infty} f(x)\, dx = 1}$$

$$\text{Mean} = \int_{-\infty}^{\infty} x \cdot \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}_{f(x)} dx$$

$$= \int_{-\infty}^{\infty} (x-\mu) \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \underbrace{\int_{-\infty}^{\infty} \mu \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}_{1 \text{ (proved above)}}$$

$= 0 + \mu$ $\qquad$ Put $x - \mu = y$

$= \underline{\underline{\mu}}$ $\qquad$ $dx = dy$

$$\frac{x-\mu}{\sqrt{2}\cdot\sigma} = t \qquad \therefore \frac{dx}{\sqrt{2}\cdot\sigma}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{t}{\sqrt{\pi}} \cdot e^{-t^2} \frac{dt}{\sqrt{2}\cdot\sigma} \qquad t^2 = y \quad t\,dt = \frac{dy}{2}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \underbrace{\int_{-\infty}^{\infty} t e^{-t^2} dt}_{0}$$

$$\text{Variance} = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$$

$$= \int t^2 \cdot e^{-t^2}\,dt$$

$$= \int t\,(te^{-t^2})\,dt$$

$$= \underline{\underline{\sigma}}$$

$$X(\underset{\text{mean}}{\mu}, \underset{\text{variance}}{\sigma^2}) \qquad f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$Y = aX + b$ → new random variable

$$E[Y] = a\,E[X] + b$$

$$\begin{aligned} \text{Var} = E[Y-E[Y]]^2 &= E[aX+\cancel{b} - aE[X] - \cancel{b}]^2 \\ &= E[aX - aE[X]]^2 \\ &= E[a(X-E[X])]^2 \\ &= a^2 E[X-E[X]]^2 \\ &= \underline{\underline{a^2\sigma^2}} \end{aligned}$$

$\because E[cX] = cE[X]$

<u>Standard</u> <u>Normal</u> <u>Distribution</u> $N(0,1)$:

Convert normal distribution into a new random variable where mean $= 0$ and variance $= 1$

$$Y = \frac{X-\mu}{\sigma} = \underbrace{\frac{1}{\sigma}}_{a} X - \underbrace{\frac{\mu}{\sigma}}_{b}$$

$$E[Y] = 0$$

$$E[Y - E[Y]]^2 = 1$$

$$E[Y] = aE[X] + b$$

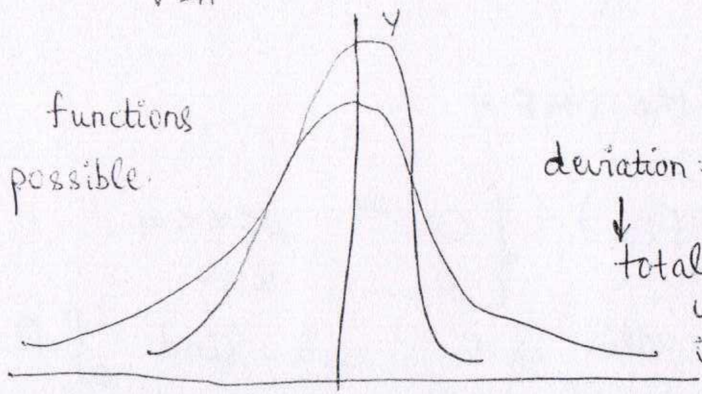
$$= \frac{1}{\sigma}\cdot\mu - \frac{\mu}{\sigma} = \underline{\underline{0}}$$

$$E[Y-E[Y]]^2 = a^2\sigma^2 = \frac{1}{\sigma^2}\cdot\sigma^2 = \underline{\underline{1}}$$

<u>RV: Y</u>

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy \quad (\because \mu = 0 \ \& \ \sigma = 1)$$

Many functions are possible.



deviation $= 1$ from 0
↓
total area under integration

$\phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-y^2} dy$ (From table - negative values not given)

$-\infty$ to $0 = 0.5$ (half of 1)
(since it is symmetric

Suppose $= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-y^2} dy$

$\phi(2.18) = 0.9854$

2.1 and 0.08
row column

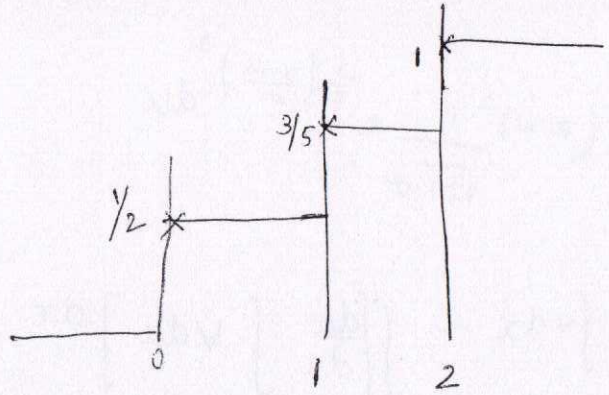
) Given:
CDF

$$F(b) = \begin{cases} 0 & b<0 \\ 1/2 & 0 \le b < 1 \\ 3/5 & 1 \le b < 2 \\ 1 & b \le 2 \end{cases}$$

Find the PMF.

2) PDF: $f(x) = \begin{cases} Ce^{-2x} & 0<x<\infty \\ 0 & x<0 \end{cases}$

Find value of C and find $\int_{-2}^{1} f(x)dx$

① 

$p(0) = 1/2$

$p(1) = 3/5 - 1/2 = \frac{6-5}{10} = \frac{1}{10}$

$p(2) = 1 - \frac{3}{5} = \frac{5-3}{5} = \frac{2}{5}$

② $\int_0^\infty Ce^{-2x} dx = 1$

$\boxed{\Rightarrow \int_{-\infty}^{\infty} f(x)dx = 0}$

$C \int_0^\infty e^{-2x} dx = 1$

$C = ?$

$C \left[\frac{e^{-2x}}{-2} \right]_0^\infty = 1$

$\frac{c}{-2} \left[e^{-\infty} - e^{-0} \right] = 1$

$\frac{c}{-2} [0-1] = 1$

$c/2 = 1$

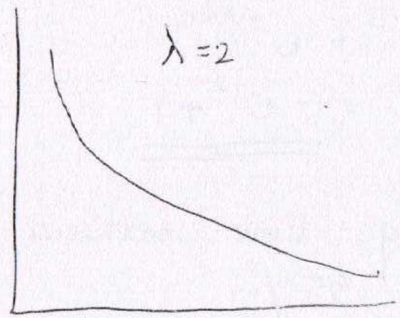
$\boxed{\therefore C = 2}$

$$\int_{-2}^{1} 2e^{-2x}\,dx = \frac{\cancel{2}}{-\cancel{2}}\left[e^{-2x}\right]_{-2}^{1}$$

$$= -1\left[e^{-2} - e^{4}\right] = \underline{\underline{e^{4} - e^{-2}}}$$

16·02·2018

Exponential Distribution :



Time elapsed between 2 events — exponentially distributed
or interarrival time

$$\text{PDF } f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \cdot \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_{0}^{\infty} x e^{-\lambda x}\,dx = \lambda\cancel{\left\{\left[\frac{x \cdot e^{-\lambda x}}{\lambda}\right]_0^{\infty} + \left[e^{-\lambda x}\frac{x^2}{2}\right]_0^{\infty}\right\}}$$

$$= \cancel{\lambda\left[\frac{-1}{\lambda}(\infty - 0)\right]} +$$ Integrating by parts we get $\underline{\underline{1/\lambda}}$

$$= \underline{\underline{\frac{1}{\lambda}}}$$

$$\text{Variance} = E[x^2] - (E[x])^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \underline{\underline{\frac{1}{\lambda^2}}}$$

$$P(X > s+t \mid X > s) = \frac{P(X > s+t) \cap P(X > s)}{P(X > s)}$$

$$= \frac{P(X > s+t)}{P(X > s)}$$

We assume X is exponentially distributed.

$$= \frac{\int_{s+t}^{\infty} \lambda e^{-\lambda x}\,dx}{\int_{s}^{\infty} \lambda e^{-\lambda x}\,dx} = \frac{\cancel{\lambda}\left[\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_{s+t}^{\infty}}{\cancel{\lambda}\left[\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_{s}^{\infty}}$$

$$= \frac{e^{-\infty\lambda} - e^{-\lambda(s+t)}}{e^{-\infty\lambda} - e^{-\lambda s}} = \frac{0 - e^{-\lambda(s+t)}}{0 - e^{-\lambda s}}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = \underline{\underline{e^{-\lambda t}}}$$

$$P(x>t) = \int_t^{\infty} \lambda e^{-\lambda x} dx = \cancel{\lambda}\left[\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_t^{\infty}$$

$$= -\left[e^{-\lambda\infty} - e^{-\lambda t}\right]$$

$$= -\left(0 - e^{-\lambda t}\right)$$

$$= \underline{\underline{e^{-\lambda t}}}$$

Hence,

$$P(x > s+t \mid x > s) = P(x>t) = e^{-\lambda t}$$

Memoryless property of exponential distribution

Here, in RHS $e^{-\lambda t}$ no 's' term is present.

Hence, RHS not dependent on LHS

Ex: Number of years lived = s, number of years will ~~leave~~ leave - not dependent on 's'.

Job arrives at every 15 seconds on an average.

$$\text{Mean} = \frac{1}{\lambda} = 15\ \text{sec} = \frac{15}{60} = \frac{1}{4}\ \text{min}$$

$$\therefore \underline{\underline{\lambda = 4\ \text{min}}}$$

<u>To find</u>:

Probability of waiting less than 30 seconds.

$$30\ \text{sec} = \frac{30}{60} = \frac{1}{2} = \underline{\underline{0.5\ \text{min}}}$$

$$P[t<0.5] = \int_{-\infty}^{0.5} \lambda e^{-\lambda x} dx = \int_{-\infty}^{0} \lambda e^{-\lambda x} dx + \int_0^{0.5} \lambda e^{-\lambda x} dx$$

$$= \lambda \int_0^{0.5} e^{-\lambda x} dx = \cancel{\lambda}\left[\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_0^{0.5}$$

$$= -e^{-\lambda \times 0.5} + e^{-\lambda \times 0}$$

$$= -e^{-4\times 0.5} = \underline{\underline{-e^{-2}+1}}$$

Maximum waiting time between two jobs with confidence 95%.

$\downarrow$

Probability of getting next job = 0.95

Maximum waiting time = t

$$P[t<t] = 0.95$$

$$\int_0^t \lambda e^{-\lambda x} dx = 0.95$$

$$\cancel{\lambda}\left[\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_0^t = 0.95$$

$$\Rightarrow -e^{-\lambda t} + e^{-\lambda \times 0} = 0.95$$

$$1 - e^{-4t} = 0.95$$

In a call center, calls arrive at an average 12 per hour. Find the probability that a call will occur in next 5 mins, given that already waited for 10 mins.

<u>Given</u>: $\frac{1}{\lambda} = 12/hr = \frac{12}{60} = \frac{1}{5}/min$

$\therefore \underline{\lambda = 5 \text{ min}}$ (Interarrival time)

$$P[\underbrace{T>5+10}_{t+s} / \underbrace{T>10}_{s}] = P[T>5] = \int_5^{\infty} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_5^{\infty} = 0 + e^{-\lambda \times 5} = \underline{\underline{e^{-25}}}$$
$[\because \lambda = 5]$

<u>Markov Inequality</u>: X is R.V (distribution is not known). X is non-negative
We know E[x] and variance

$$E[x] = \int_{-\infty}^{\infty} \underbrace{x f(x)\,dx}_{\text{not known}} = \int_{-\infty}^{a} x f(x)\,dx + \int_{a}^{\infty} x f(x)\,dx$$

<u>To find</u>:
$$P[x>a] = \int_a^{\infty} f(x)\,dx$$

$E[x] \geq \int_a^{\infty} x f(x)\,dx$ [Since $\int_{-\infty}^{a} x f(x)\,dx$ is +ve under curve]

$$E[x] \geq a \int_a^{\infty} f(x)\,dx$$

$$\boxed{E[x] \geq a \cdot P[x>a]}$$

$$\boxed{\therefore P[x>a] \leq \frac{E[x]}{a}}$$

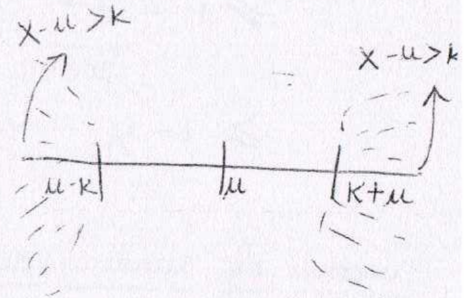
<u>20.02.2018</u>

<u>Chebyshev inequality</u>: X is non-negative R.V with μ, σ

$P\{|X-\mu| \geq k\} \leq \sigma^2/k^2$

$\Downarrow$ both are same

$P\{(X-\mu)^2 \geq k^2\}$



Apply Markov inequality

$$\leq \frac{E[(x-\mu)^2]}{k^2}$$

$$\leq \frac{E[x-\mu]^2}{k^2} \leq \underline{\underline{\frac{\sigma^2}{k^2}}}$$

Post office handles on an average 1000 letters per day Variance = 2000. ① What is the probability that post office will handle more than 2000 letters tomorrow?

(RV)

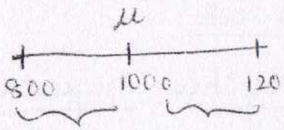
X: Denotes no of letters handled by the post office

$P\{X>2000\} \rightarrow$ To find

$P\{X>\underbrace{2000}_{a}\} \leq \frac{E[X]}{2000} \leq \frac{1000}{2000} \leq \underline{\underline{\frac{1}{2}}}$

2) Probability that post office will handle letters b/w 800 and 1200.

$P\{|X-1000| \leq 200\}$



$= 1 - P\{|X-1000| \geq 200\}$

$\geq 1 - \frac{\sigma^2}{200^2} \geq 1 - \frac{2000}{200^2} \geq 1 - \frac{2\cancel{000}}{2\cancel{00} \times 2\cancel{00}}$

$\geq 1 - 1/10 \geq \underline{\underline{9/10}}$

Function of random variable:

Discrete:

Given PMF of $X \rightarrow$ To find PMF of $f(x)$ (Ex: X^2, $X+1$)

Ex: $p_X(0) = 0.2$, $p_X(1) = 0.5$, $p_X(2) = 0.3 \rightarrow$ 3 discrete points

To find: PMF of X^2 ($Y = X^2$)

$X = \{0, 1, 2\}$ $\quad Y = \{0, 1, 2^2 = 4\}$

$p_Y(0) = 0.2, \quad p_Y(1) = 0.5, \quad p_Y(4) = 0.3$

$Y: S \rightarrow \{0, 1, 4\}$

Mean $= \frac{0\times0.2 + 1\times0.5 + 4\times0.3}{0.2+0.5+0.3} = \frac{0.5+1.2 = 1.7}{1}$

Continuous:

Given PDF of X, find out PDF of $f(x) = X^2$

$Y = X^2$

Find CDF for $Y = P[Y \leq a] = F(a) = \int_{-\infty}^{a} f(x)\,dx$

Then, PDF for $Y = f(a) = \frac{d}{da}[F(a)]$

Let X be a RV with PDF $f(x) = \begin{cases} 3/2\, x^2 & -1 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$

Find the PDF of X^2.

Let $X^2 = Y$

CDF for $Y = F(a) = P[Y \leq a] = P[X^2 \leq a]$

$= P\{-\sqrt{a} \leq X \leq 0 \cup 0 \leq X \leq \sqrt{a}\}$

$= P\{-\sqrt{a} \leq X \leq 0\} + P\{0 \leq X \leq \sqrt{a}\}$

$= \int_{-\sqrt{a}}^{0} \frac{3}{2} x^2\,dx + \int_{0}^{\sqrt{a}} \frac{3}{2} x^2\,dx$

Case 1: Let $0 < a < 1$

$= \frac{\cancel{3}}{2} \cdot \left[\frac{x^3}{\cancel{3}}\right]_{-\sqrt{a}}^{0} + \frac{\cancel{3}}{2}\left[\frac{x^3}{\cancel{3}}\right]_{0}^{\sqrt{a}}$

$= \frac{1}{2}[0^3 - (-\sqrt{a})^3] + \frac{1}{2}[(\sqrt{a})^3 - 0^3]$

$$= \tfrac{1}{2} \times a^{3/2} + \tfrac{1}{2} \times a^{3/2} = a^{3/2}$$

$$\therefore \; \underline{\underline{F(a) = a^{3/2}}}$$

PDF $f_Y(a) = \frac{d}{da}\left[F(a)\right] = \frac{d}{da}\left(a^{3/2}\right)$

$$= \frac{3}{2} a^{3/2 - 1} = \underline{\underline{\frac{3}{2} a^{1/2}}} \qquad 0 < a < 1$$

$\downarrow$ negative values not allowed for $\sqrt{a}$

<u>Case 2</u> : $a > 1$

$$= \int_{-1}^{0} \frac{3}{2} x^2 dx + \int_{0}^{1} \frac{3}{2} x^2 dx$$

$$= \frac{\cancel{3}}{2}\left[\frac{x^3}{\cancel{3}}\right]_{-1}^{0} + \frac{\cancel{3}}{2}\left[\frac{x^3}{\cancel{3}}\right]_{0}^{1}$$

$$= \frac{1}{2}\left[0 - (-1)^3\right] + \frac{1}{2}\left[1^3 - 0\right]$$

$$F(a) = \tfrac{1}{2} + \tfrac{1}{2} = \underline{\underline{1}}$$

$$f_Y(a) = \frac{d}{da}\left[F(a)\right] = \frac{d}{da}(1) = \underline{\underline{0}} \qquad a > 1$$

$$f_Y(a) = \begin{cases} \tfrac{3}{2} a^{1/2} & 0 < a < 1 \\ 0 & a > 1 \end{cases}$$

<u>Proposition</u> : (To find mean)

X : discrete $\quad E[g(x)] = \sum g(x)\, \mathrm{PMF}(x)$

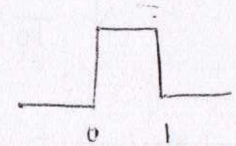
No need to find PMF of $g(x)$ to find $E[g(x)]$

X : continuous $\quad E[g(x)] = \int g(x) \underbrace{f(x)}_{\text{PDF of } X} dx$

<u>21·02·2018</u>

X is uniform RV over $(0,1)$. Find the PDF of X^3.

PDF : $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$



Let $X^3 = Y$

CDF $= F(a) = P[Y \le a] = P[X^3 \le a] = P[X \le a^{1/3}]$

$$= \int_{0}^{a^{1/3}} 1\, dx = \left[x\right]_{0}^{a^{1/3}} = \underline{\underline{a^{1/3}}}$$

$$\text{PDF} = \frac{d}{da}\left(F(a)\right) = \frac{d}{da}\left(a^{1/3}\right)$$

$$f(a) = \frac{1}{3} a^{1/3 - 1} = \underline{\underline{\frac{a^{-2/3}}{3}}}$$

$$f_Y(a) = \begin{cases} \frac{1}{3} a^{-2/3} & 0 < a < 1 \\ 0 & \text{otherwise} \end{cases}$$

or

$$f_Y(x) = \begin{cases} \frac{1}{3} x^{-2/3} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

<u>JOINT</u> <u>DISTRIBUTED</u> <u>RV</u> : X and Y

<u>Discrete</u> : $X \,\&\, Y \rightarrow RV$

Joint PMF : $P(X = x, Y = y) = p(x, y)$

$$p_X(x) = \sum_{y} p(x, y)$$

↓ To get value of $\underbrace{X = x}_{1 \text{ value}}$, sum all '$\underbrace{y}_{\text{all values}}$' values

$$p_Y(y) = \sum_{x} p(x, y)$$

<u>Cummulative</u> :

$F(a, b) = P(X \le a, Y \le b)$

$F_X(a) = P\{X \le a\} \rightarrow$ Y can be anything

$= P\{X \le a, Y \le \infty\}$

$F_Y(b) = P\{Y \le b\}$

$= P\{X \le \infty, Y \le b\}$

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406117005 - K. Sai prabanjan Kumar
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406117001 - Randive Kunal Dhanraj

1. Cryptography Theory & Practice - 3rd edition
Douglas R. Stinson

2. Wen Bo Mao.

3. Cryptography & N/w Security -
Behrouz A. Forouzan & Debdeep Mukhopadhyay

4. William Stallings

1 2 3
1 3 2

1 way

2 1 3
2 3 1
3 1 2
3 2 1

1 2 3

ordered