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Review of systems:

Review of first order system:

Examples:

1. Liquid tank system

$$A \frac{dh'}{dt} = F_i' - F'$$

Change of height of the tank (output) and change of inlet flow rate (input) related by a first order differential equation.

2. Thermometer & thermocouple

$$mC_p \frac{dT'}{dt} = hA(T_a' - T')$$

Change of thermometer temperature (output) and Change of environment temperature (input) related by a first order differential equation.

3. Mixing process

$$\frac{d(Vy')}{dt} = qx' - qy'$$

Change of solute outlet concentration (output) and Change of inlet concentration (input) related by a first order differential equation.

4. Stirred Tank Reactor

Aim:

How output varies with time when system experienced change in input.

Definition: Any system whose input and output are related by a first order differential equation is called first order system.

General form of first order system:

$$a \frac{dy'}{dt} + by' = cf'(t)$$

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Where, $y' = y - y_s$ and $f'(t) = f - f_s$

We need to know how y and f related each other

Apply Laplace transform to get relation between y and f

$$\left(\frac{a}{b}\right) \frac{dy'}{dt} + y' = \left(\frac{c}{b}\right) f'(t)$$

$$\tau_p \frac{dy'}{dt} + y' = k_p f'(t)$$

Apply Laplace transform

$$\tau_p [s\bar{y}(s) - y'(0)] + \bar{y}(s) = k_p \bar{f}(s)$$

$y'(0) = 0$ [at steady state ($t = 0$) change of y and derivative of change of y is zero]

$$\frac{\bar{y}(s)}{\bar{f}(s)} = \frac{k_p}{\tau_p s + 1} = G(s), \text{ Where } G(s) \text{ is transfer function}$$

Transfer function: Transfer function is defined as the ratio between output and input in Laplace domain

$f'(t)$ is the change of input. Convert $f'(t)$ from time domain to Laplace domain and find $y'(s)$

$$\bar{y}(s) = \frac{k_p}{\tau_p s + 1} \bar{f}(s)$$

Apply inverse of Laplace and find $y'(t)$ [Change of y in time domain]

$f'(t)$	$\bar{f}(s)$
Unit impulse $\delta(t)$	1
Unit pulse $u(t)$	$1/s$
Linear $t u(t)$	$1/s^2$
Quadratic $t^2 u(t)$	$2/s^3$
Exponential $e^{-at} u(t)$	$1/(s+1)$
Sin $\sin(at) u(t)$	$a/(s^2+a^2)$
Cos $\cos(at) u(t)$	$s/(s^2+a^2)$

If $f'(t) = B u(t)$

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$$\bar{y}(s) = \frac{k_p}{(\tau_p s + 1)} \frac{B}{s}$$

$$\bar{y}(s) = Bk_p \left(\frac{1}{s} - \frac{1}{s + 1/\tau_p} \right)$$

$$y(t) = Bk_p \left(1 - e^{-\frac{t}{\tau_p}} \right) \quad t \geq 0$$

Special case:

If $b = 0$;

$$a \frac{dy'}{dt} = cf'(t)$$

$$\tau_p \frac{dy'}{dt} = k_p f'(t)$$

$$\frac{\bar{y}(s)}{\bar{f}(s)} = \frac{k_p}{\tau_p s}$$

$$\bar{y}(s) = \frac{k_p}{\tau_p s} \bar{f}(s)$$

If $f'(t) = B$

$$\bar{y}(s) = \frac{k_p}{\tau_p s} \frac{B}{s}$$

$$y(t) = Bk_p t$$

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Thermometer:

Assumptions:

1. All the resistance to heat transfer resides in the film surrounding the bulb (i.e., the resistance offered by the glass and mercury is neglected).
2. All the thermal capacity is in the mercury. Furthermore, at any instant the mercury assumes a uniform temperature throughout.
3. The glass wall containing the mercury does not expand or contract during the transient response.

Apply energy balance in the thermometer,

The rate of flow of heat through the film resistance surrounding the bulb causes the internal energy of the mercury to increase at the same rate.

$$mC_p \frac{dT}{dt} = hA(T_a - T)$$

$T_a = \text{Environment temperature and } T = \text{Thermometer reading,}$

$A = \text{surface area of bulb for heat transfer}$

h depends on surrounding fluid flow and properties and bulb dimension.

At steady state, $0 = mC_p \frac{dT_s}{dt} = hA(T_{as} - T_s)$

In deviation form,

$$mC_p \frac{dT'}{dt} = hA(T_a' - T')$$

$$\left(\frac{mC_p}{hA}\right) \frac{dT'}{dt} = (T_a' - T')$$

$$\left(\frac{mC_p}{hA}\right) \frac{dT'}{dt} + T' = T_a'$$

$$\tau_p \frac{dT'}{dt} + T' = k_p T_a'$$

Where, $\tau_p = \frac{mC_p}{hA}$ and $k_p = 1$

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$$\bar{T}(s) = \frac{k_p}{\tau_p s + 1} \bar{T}_a(s)$$

$$\frac{\bar{T}(s)}{\bar{T}_a(s)} = \frac{k_p}{\tau_p s + 1} = G(s)$$

$G(s)$ = transfer function of the system. It is ratio of the Laplace transform of the deviation in the thermometer reading to the Laplace transform of the deviation in the surrounding temperature. It relates the cause and effect.

If $T_a'(t) = Bu(t)$

$$\bar{T}(s) = \frac{1}{(\tau_p s + 1)} \frac{B}{s}$$

$$\bar{T}(s) = B \left(\frac{1}{s} - \frac{1}{s + 1/\tau_p} \right)$$

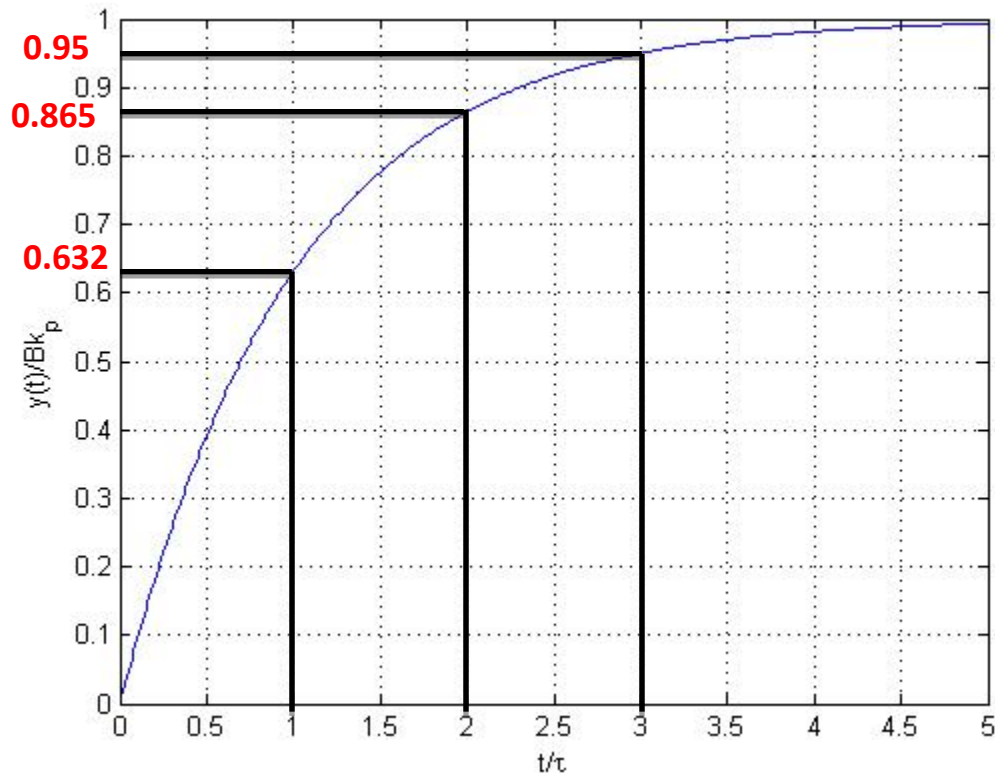
$$T(t) = B \left(1 - e^{-\frac{t}{\tau_p}} \right) \quad t \geq 0$$

Note: $\frac{y(t)}{Bk_p} = \left(1 - e^{-\frac{t}{\tau_p}} \right) \quad t \geq 0$

Slope of graph $\frac{y(t)}{Bk_p}$ vs $\frac{t}{\tau_p}$ is $\frac{d\left(\frac{y(t)}{Bk_p}\right)}{d\left(\frac{t}{\tau_p}\right)} = e^{-\frac{t}{\tau_p}}$

Initial slope $\frac{d\left(\frac{y(t)}{Bk_p}\right)}{d\left(\frac{t}{\tau_p}\right)} \Big|_{t=0} = e^{-\frac{t}{\tau_p}} \Big|_{t=0} = 1$

Plot $y(t)$ vs t/τ :



Note:

1. At $t = \tau$, $y = 0.632Bk_p$

2. If the initial rate of change of output were maintained, the response would be completed in one time constant.

$$\frac{y(t)}{Bk_p} = \left(1 - e^{-\frac{t}{\tau_p}}\right)$$

Take derivative w.r.t. time

$$\frac{d}{dt} \left[\frac{y(t)}{Bk_p} \right] \text{ at } t = 0 \text{ is } \frac{1}{\tau_p}$$

Straight line with slope $\frac{1}{\tau_p}$ and originates from 0, $P = \frac{1}{\tau_p} t$

P is 1 when $t = \tau_p$

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Significance of time constant and steady state gain

Higher time constant – slower response (reaches new steady state faster)

At new steady state change of output is equal to change of input

Problem 1. A thermometer having time constant of 0.1 min is at steady state temperature of 90 °C. At time $t = 0$, the thermometer is placed in a temperature bath maintained at 100 °C. Determine the time needed for the thermometer to read 98 °C.

Solution:

Time constant $\tau_p = 0.1$ min. Steady state temperature $T_s = 90$ °C.

Sudden change of temperature bath = change in input (step Input) $B = 100 - 90 = 10$ °C.

$K_p = 1$.

$$T(t) = Bk_p \left(1 - e^{-\frac{t}{\tau_p}} \right)$$

Need to find time when temperature rises from 90 to 98 °C.

Hence $T(t) = 98 - 90 = 8$ °C

Therefore, $8 = (10)(1)(1 - \exp(-t/0.1))$

$t = 0.161$ min

Problem 2. A thermometer of time constant 10 s, initially at 30 °C, is suddenly immersed into water bath at 100 °C. How long will it take for the thermometer reading to reach 90 °C.

Solution:

We need to determine time when thermometer reading reaches 90 °C.

Hence $T(t) = 90 - 30 = 60$ °C

$B = 100 - 30 = 70$ °C

Therefore, $60 = (70)(1)(1 - \exp(-t/10))$

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$$t = 19.5 \text{ s.}$$

Problem 3. When a thermometer at 30°C is placed in a water bath at 90°C , the initial rate of rise in thermometer temperature is found to be 2°C/s . What is the time constant of the thermometer?

What will thermometer read after one minute?

Solution:

$$B = 60^\circ\text{C}$$

$$T(t) = B \left(1 - e^{-\frac{t}{\tau_p}} \right)$$

$$\frac{dT}{dt} \Big|_{t=0} = (B/\tau_p) e^{-t/\tau_p} \Big|_{t=0} = \frac{B}{\tau_p} = 2$$

$$(60/\tau_p)(1) = 2$$

$$\tau_p = 30 \text{ s}$$

After one minute (60 s) temperature

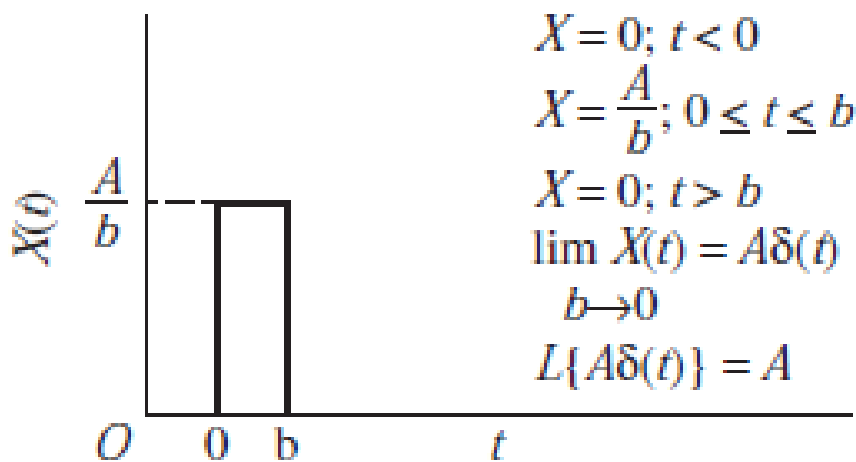
$$T = 30 + (60)(1 - \exp(-60/30)) = 81.9^\circ\text{C}$$

Response of first order system with impulse input:

Impulse function: Mathematically, the impulse function of magnitude A is defined as

$$X(t) = A\delta(t)$$

Where, $A\delta(t)$ is the unit impulse function



A first order system can be expressed as

$$Y(s) = \frac{k_p}{(\tau_p s + 1)} X(s)$$

With the application of impulse input of magnitude A

$$Y(s) = \frac{k_p/\tau_p}{(s + 1/\tau_p)} A$$

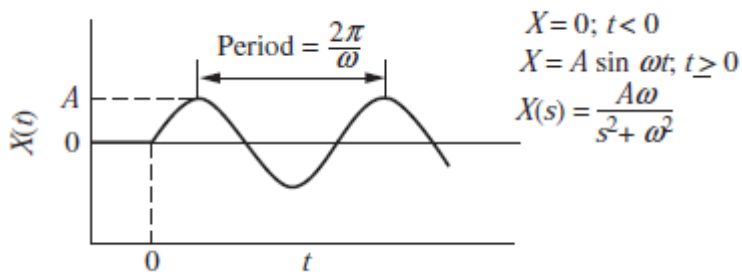
$$Y(s) = \frac{Ak_p}{\tau_p} e^{-t/\tau_p}, \text{ Response decreases exponentially}$$

Response of first order system with sinusoid input:

The function is represented mathematically by the equation

$$X = 0, \quad t < 0$$

$$X = A \sin \omega t, \quad t \geq 0$$



$$Y(s) = \frac{k_p}{(\tau s + 1)} X(s)$$

$X(s) = \frac{A\omega}{s^2 + \omega^2}$, where, A is amplitude, ω is frequency in radian, $\omega = 2\pi f$

f = frequency (cycles per time)

$$Y(s) = \frac{k_p/\tau}{(s + 1/\tau)} \frac{A\omega}{s^2 + \omega^2}$$

This equation can be solve for Y(t) by means of partial fraction. The result is

$$Y(t) = \frac{A\omega\tau e^{-t/\tau}}{\tau^2\omega^2 + 1} - \frac{A\omega\tau}{\tau^2\omega^2 + 1} \cos \omega t + \frac{A}{\tau^2\omega^2 + 1} \sin \omega t$$

The equation can be written in another form applying the trigonometric identity

$$Y(t) = \frac{A\omega\tau}{\tau^2\omega^2 + 1} e^{-t/\tau} + \frac{A}{\sqrt{\tau^2\omega^2 + 1}} \sin(\omega t + \phi)$$

Where,

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$$\phi = \tan^{-1}(-\omega\tau)$$

As $t \rightarrow \infty$, the first term on the right side of the equation vanishes and leaves only the ultimate periodic solution, which is sometimes called the steady state solution

$$Y(t)|_s = \frac{A}{\sqrt{\tau^2\omega^2 + 1}} \sin(\omega t + \phi)$$

Comparing the result with the input function, we see that

1. The output is a sine wave with a frequency ω equal to that of the input signal.
2. The ratio of output amplitude to input amplitude is $1/\sqrt{\tau^2\omega^2 + 1}$. This ratio is always smaller than 1. We often state this by saying that the signal is *attenuated*.
3. The output lags behind the input by an angle ϕ . It is clear that lag occurs, for the sign of ϕ is always negative.*

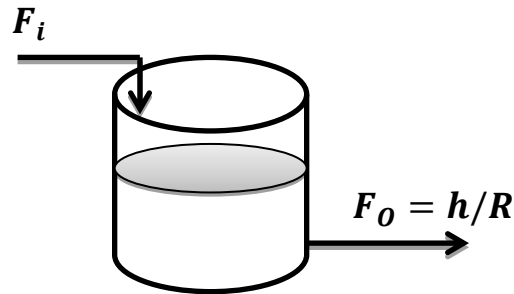
Response of first order system with Pulse input: (Scan)

Response of first order system with Ramp input: (Scan)

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Liquid tank system:

Assumptions: Constant liquid density, Outlet flow is linearly varies with tank level.



Apply mass balance,

$$A \frac{dh}{dt} = F_i - h/R$$

At steady state,

$$0 = A \frac{dh_s}{dt} = F_{is} - h_s/R$$

$$h' = h - h_s \quad \text{and} \quad F_i'(t) = F_i - F_{is}$$

$$A \frac{dh'}{dt} = F_i' - h'/R$$

$$A \frac{dh'}{dt} + h'/R = F_i'$$

Multiply both sides by 'R'

$$AR \frac{dh'}{dt} + h' = RF_i'$$

$$\tau_p \frac{dh'}{dt} + h' = k_p F_i'$$

Where, $\tau_p = AR$ **and** $k_p = R$

$$\bar{h}(s) = \frac{k_p}{\tau_p s + 1} \bar{F}_i(s)$$

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If $F_i'(t) = Bu(t)$

$$\bar{h}(s) = \frac{R}{(ARs + 1)} \frac{B}{s}$$

$$\bar{h}(s) = BR \left(\frac{1}{s} - \frac{1}{s + 1/\tau_p} \right)$$

$$h'(t) = BR \left(1 - e^{-\frac{t}{\tau_p}} \right) \quad t \geq 0$$

The ultimate change in $h'(t)$ for a unit change in input is 'R'

$$\bar{h}(s) = BR$$

R is simply the conversion factor that relates $h(t)$ and $F_0(t)$ when the system is at steady state.

This R is called steady state gain. This is the ratio between change of output and change of input at steady state.

The above system is a self-regulating system because the output is limited for a sustained change in input. For a sustained change in input the height will change with time till inlet and outlet flow become same.

Relate inlet and outlet flow

$$\bar{h}(s) = \frac{R}{\tau_p s + 1} \bar{F}_i(s)$$

$$\text{Substitute } \bar{h}(s) = R \bar{F}_o(s)$$

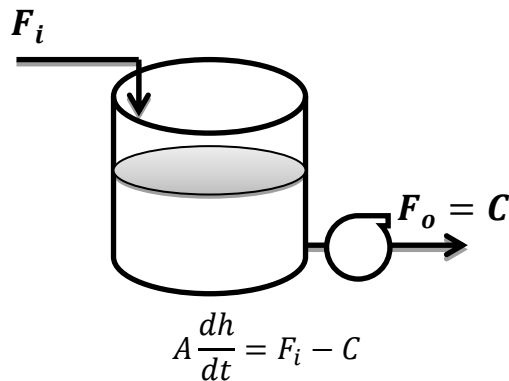
$$\bar{F}_o(s) = \frac{1}{\tau_p s + 1} \bar{F}_i(s)$$

If $F_i'(t) = Bu(t)$

$$\bar{F}_o(s) = B \left(1 - e^{-\frac{t}{\tau_p}} \right) \quad t \geq 0$$

Here, $K_p=1$ (dimensionless) which is expected because the input and output variable have the same unit.

Special Case:



At steady state,

$$A \frac{dh_s}{dt} = F_{is} - C$$

$$h' = h - h_s \quad \text{and} \quad F_i'(t) = F_i - F_{is}$$

$$A \frac{dh'}{dt} = F_i' - 0$$

$$A \frac{dh'}{dt} = F_i'$$

$$A \frac{dh'}{dt} = F_i'$$

$$\bar{y}(s) = \frac{1}{As} \bar{F}_i(s)$$

If $F_i'(t) = Bu(t)$

$$\bar{y}(s) = \frac{1}{As} \frac{B}{s}$$

$$y(t) = Bt$$

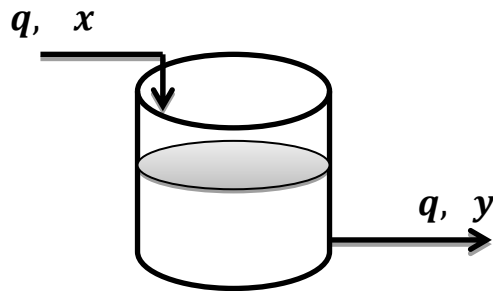
Here response grows without limit for a sustained change in input. Such a system called non-regulating system. A first order self-regulating system with infinite resistance behaves as non-regulating system.

$$\bar{h}(s) = \frac{R}{(ARs + 1)} \frac{B}{s}$$

If $R \rightarrow \infty$

$$\bar{h}(s) = \frac{R}{ARs} \frac{B}{s} = \frac{1}{As} \frac{B}{s}$$

Mixing process:



Assumptions:

Well mixed system

Constant hold up

Inlet flow rate = outlet flow rate = q (volumetric rate)

Mole balance

$$qx - qy = V \frac{dy}{dt}$$

At steady state

$$qx_s - qy_s = V \frac{dy_s}{dt}$$

In terms of deviation

$$\frac{d(Vy')}{dt} = qx' - qy'$$

Assume $V = \text{constant}$

$$V \frac{d(y')}{dt} = qx' - qy'$$

$$\bar{y}(s) = \frac{k_p}{\tau_p s + 1} \bar{x}(s)$$

Where, $\tau_p = V/q$ and $k_p = 1$

Linearization (Scan)

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Stirred Tank Reactor (STR)

Assumptions:

Well mixed system

Constant hold up

Inlet flow rate = outlet flow rate = q

First order reaction

Mole balance

$$qC_{Ai} - qC_A - VkC_A = V \frac{dC_A}{dt}$$

At steady state

$$qC_{Ais} - qC_{As} - VkC_{As} = V \frac{dC_{As}}{dt}$$

In terms of deviation

$$q\bar{C}_{Ai} - q\bar{C}_A - Vk\bar{C}_A = V \frac{d\bar{C}_A}{dt}$$

Taking Laplace transform

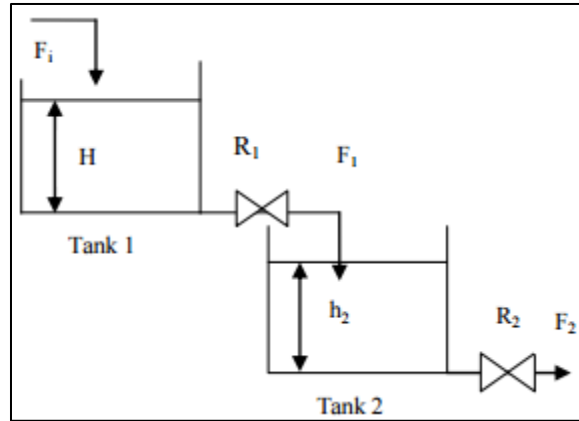
$$\frac{\bar{C}_A(s)}{\bar{C}_{Ai}(s)} = \frac{q}{Vs + (q + Vk)}$$

$$\frac{\bar{C}_A(s)}{\bar{C}_{Ai}(s)} = \frac{k'}{\tau s + 1}, \text{ where } k' = \frac{q}{q + Vk} \quad \tau = \frac{V}{q + kV}$$

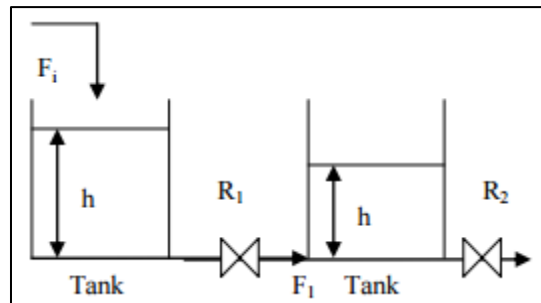
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First order system in series:

Non - interacting:



Interacting:



Non - interacting:

Apply mass balance in 1st tank at any time t -

$$A_1 \frac{dh_1}{dt} = F_i - \frac{h_1}{R_1}, \text{ outlet flow } F = \frac{h_1}{R_1}$$

At steady state

$$A_1 \frac{dh_{1s}}{dt} = F_{is} - h_{1s}/R_1$$

In deviation form, above equation can be written as follow

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$$A_1 \frac{dh'_1}{dt} = F'_i - \frac{h'_1}{R_1}, \quad \text{Where, } h'_1 = h_1 - h_{1s} \text{ and } F'_i = F_i - F_{is}$$

Apply Laplace transform both side

$$A_1[s\bar{h}_1(s) - h'_1(0)] + \frac{\bar{h}_1(s)}{R_1} = \bar{F}_i(s)$$

$$(A_1 R_1 s + 1)\bar{h}_1 = R_1 \bar{F}_i(s)$$

$$\bar{h}_1 = \frac{R_1}{(A_1 R_1 s + 1)} \bar{F}_i(s)$$

Similarly, apply mass balance in 2nd tank at any time t -

$$A_2 \frac{dh'_2}{dt} = \frac{h'_1}{R_1} - \frac{h'_2}{R_2}, \quad \text{Where, } h'_2 = h_2 - h_{2s}$$

Apply Laplace transform both side

$$A_2[s\bar{h}_2(s) - h'_2(0)] + \frac{\bar{h}_2(s)}{R_2} = \frac{\bar{h}_1(s)}{R_1}$$

$$(A_2 R_2 s + 1)\bar{h}_2 = \frac{R_2}{(A_1 R_1 s + 1)} \bar{F}_i(s)$$

$$\bar{h}_2 = \frac{R_2}{(A_1 R_1 s + 1)(A_2 R_2 s + 1)} \bar{F}_i(s)$$

$$\bar{h}_2 = \frac{R_2}{(A_1 R_1 A_2 R_2)s^2 + (A_1 R_1 + A_2 R_2)s + 1} \bar{F}_i(s)$$

$$\bar{h}_2 = \frac{R_2}{(\tau_1 \tau_2)s^2 + (\tau_1 + \tau_2)s + 1} \bar{F}_i(s)$$

$$\bar{h}_2 = \frac{R_2}{(\tau^2 s^2 + 2\tau\zeta s + 1)} \bar{F}_i(s)$$

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Where, $\tau = \sqrt{\tau_1 \tau_2}$ and $\zeta = \frac{(\tau_1 + \tau_2)}{2\sqrt{\tau_1 \tau_2}} = \frac{\text{Arithmetic mean of } \tau_1 \text{ and } \tau_2}{\text{Geometric mean of } \tau_1 \text{ and } \tau_2} > 1$

Case 1:

$$\tau_1 \neq \tau_2$$

$$\bar{h}_2(s) = \frac{R_2}{(\tau_1 s + 1)(\tau_2 s + 1)} \bar{F}_i(s)$$

For a step change in input,

$$\bar{h}_2(s) = \frac{R_2}{(\tau_1 s + 1)(\tau_2 s + 1)} \frac{B}{s} = \frac{C_1}{s} + \frac{C_2}{(\tau_1 s + 1)} + \frac{C_3}{(\tau_2 s + 1)}$$

Interacting Tank:

Apply mass balance in tank 1 and express equation in deviation form

$$A_1 \frac{dh'_1}{dt} = F'_i - \frac{(h'_1 - h'_2)}{R_1}$$

$$A_1 R_1 \frac{dh'_1}{dt} + h'_1 = R_1 F'_i + \mathbf{h'_2}$$

Apply Laplace transform both side

$$A_1 R_1 (s \bar{h}_1(s) - h'_1(0)) + \bar{h}_1(s) = R_1 \bar{F}_i(s) + \mathbf{\bar{h}_2(s)}$$

$$(A_1 R_1 s + 1) \bar{h}_1(s) = R_1 \bar{F}_i(s) + \mathbf{\bar{h}_2(s)}$$

$$\bar{h}_1(s) = \frac{R_1 \bar{F}_i(s) + \mathbf{\bar{h}_2(s)}}{(A_1 R_1 s + 1)}$$

Apply mass balance in tank 2 and express in terms of deviation form

$$A_2 \frac{dh'_2}{dt} = \frac{(h'_1 - h'_2)}{R_1} - \frac{h'_2}{R_2}$$

Multiply both sides by $R_1 R_2$

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$$A_2 R_1 R_2 \frac{dh'_2}{dt} = R_2 (h'_1 - h'_2) - R_1 h'_2$$

$$A_2 R_1 R_2 \frac{dh'_2}{dt} + R_1 h'_2 + R_2 h'_2 = R_2 h'_1$$

Apply Laplace transform both sides

$$A_2 R_1 R_2 [s\bar{h}_2(s) - h'_2(0)] + (R_1 + R_2)\bar{h}_2 = R_2 \bar{h}_1$$

Substituted $\bar{h}_1$

$$(A_2 R_1 R_2 s + R_1)\bar{h}_2 + R_2 \bar{h}_2 = R_2 \frac{R_1 \bar{F}_i(s) + \bar{h}_2(s)}{(A_1 R_1 s + 1)}$$

Multiply both sides by $(A_1 R_1 s + 1)$

$$R_1 (A_2 R_2 s + 1)(A_1 R_1 s + 1)\bar{h}_2 + R_2 \bar{h}_2 (A_1 R_1 s + 1) = R_1 R_2 \bar{F}_i(s) + R_2 \bar{h}_2(s)$$

$$R_1 (A_2 R_2 s + 1)(A_1 R_1 s + 1)\bar{h}_2 + A_1 R_1 R_2 s \bar{h}_2 + R_2 \bar{h}_2 = R_1 R_2 \bar{F}_i(s) + R_2 \bar{h}_2(s)$$

Divide both sides by R_1

$$(A_2 R_2 s + 1)(A_1 R_1 s + 1)\bar{h}_2 + A_1 R_2 s \bar{h}_2 = R_2 \bar{F}_i(s)$$

$$\bar{h}_2 = \frac{R_2}{(A_2 R_2 s + 1)(A_1 R_1 s + 1) + A_1 R_2 s} \bar{F}_i(s)$$

$$\bar{h}_2 = \frac{R_2}{A_1 R_1 A_2 R_2 s^2 + (A_2 R_2 + A_1 R_1 + A_1 R_2)s + 1} \bar{F}_i(s)$$

$$\bar{h}_2 = \frac{1}{(\tau^2 s^2 + 2\tau\zeta s + 1)} \bar{F}_i(s)$$

Where, $\tau = \sqrt{A_1 R_1 A_2 R_2}$ and $\zeta = \frac{(A_2 R_2 + A_1 R_1 + A_1 R_2)}{2\sqrt{A_1 R_1 A_2 R_2}} > 1$

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Response of Non- interacting system with Step input

For same time constant and different time constant

Response of Interacting system with Step input

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Review of Second order system:

Examples:

1. U tube manometer

$$\frac{L}{2g} \frac{d^2 h}{dt^2} + \frac{4\mu L}{\rho g R^2} \frac{dh}{dt} + h = \frac{1}{2\rho g} \Delta P$$

$$\tau^2 \frac{d^2 h}{dt^2} + 2\zeta\tau \frac{dh}{dt} + h = k_p \Delta P$$

2. Damped vibrator

Definition:

Any system that can be represented by a second order differential equation is a second order system.

General form of second order system:

$$a \frac{d^2 y'}{dt^2} + b \frac{dy'}{dt} + cy' = df'(t)$$

$$\left(\frac{a}{c}\right) \frac{d^2 y'}{dt^2} + \left(\frac{b}{c}\right) \frac{dy'}{dt} + y' = \left(\frac{d}{c}\right) f'(t)$$

$$\tau^2 \frac{d^2 y'}{dt^2} + 2\zeta\tau \frac{dy'}{dt} + y' = k_p f'(t)$$

$$\tau^2 \left[s^2 \bar{y}(s) - sy'(0) - \frac{dy'(0)}{dt} \right] + 2\zeta\tau [s\bar{y}(s) - y'(0)] + \bar{y}(s) = k_p \bar{f}(s)$$

$$\frac{\bar{y}(s)}{\bar{f}(s)} = \frac{k_p}{\tau^2 s^2 + 2\zeta\tau s + 1} = G(s)$$

ζ is damping factor

Response of Second order system:

Step response:

$$\bar{y}(s) = \frac{k_p}{\tau^2 s^2 + 2\zeta\tau s + 1} \bar{f}(s)$$

If $f'(t) = B$

$$\bar{y}(s) = \frac{k_p}{\tau^2 s^2 + 2\zeta\tau s + 1} \frac{B}{s}$$

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$$\bar{y}(s) = \frac{k_p/\tau^2}{s^2 + 2\zeta s/\tau + 1/\tau^2} \frac{B}{s}$$

$$\bar{y}(s) = \frac{k_p/\tau^2}{(s-p)(s-q)} \frac{B}{s}$$

Where, $p = -\frac{\zeta}{\tau} + \frac{\sqrt{\zeta^2-1}}{\tau}$, $q = -\frac{\zeta}{\tau} - \frac{\sqrt{\zeta^2-1}}{\tau}$

Case 1: $\zeta > 1$, Over damped system, roots are **complex and positive**

$$y'(t) = 1 - e^{-\zeta t/\tau} \left(\cosh \frac{\sqrt{\zeta^2-1}}{\tau} t + \frac{\zeta}{\sqrt{\zeta^2-1}} \sinh \frac{\sqrt{\zeta^2-1}}{\tau} t \right)$$

Case 2: $\zeta = 1$, Critically damped system, roots are **real and equal**

$$y'(t) = 1 - e^{-\zeta t/\tau} (1 + t/\tau)$$

Case 3: $\zeta < 1$, under damped system, roots are **complex and negative**

$$p = -\frac{\zeta}{\tau} + i \frac{\sqrt{1-\zeta^2}}{\tau}, \quad q = -\frac{\zeta}{\tau} - i \frac{\sqrt{1-\zeta^2}}{\tau}$$

Taking Laplace transform,

$$y'(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta t/\tau} \sin \left(\frac{\sqrt{1-\zeta^2}}{\tau} t + \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\tau} \right)$$

Derivation of response of second order system with step input:

$$\bar{y}(s) = \frac{k_p/\tau^2}{(s-p)(s-q)} \frac{B}{s}$$

Case 1: $\zeta > 1$, Over damped system, roots are **complex and positive**

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$$\bar{y}(s) = \frac{k_p/\tau^2}{(s+a)(s+b)} \frac{B}{s}$$

Let, $\frac{1}{\tau} = w$, $a = -p$, $b = -q$

$$a = w\zeta - w\sqrt{\zeta^2 - 1}, \quad b = w\zeta + w\sqrt{\zeta^2 - 1}$$

Therefore,

$$\bar{y}(s) = \frac{w^2 k_p}{(s+a)(s+b)} \frac{B}{s}$$

Note: Laplace transform of $\frac{K}{s(s+a)(s+b)} = \frac{K}{ab} \left[1 - \frac{be^{-at} - ae^{-bt}}{b-a} \right]$

$$a \cdot b = w^2 (\zeta - \sqrt{\zeta^2 - 1}) \cdot (\zeta + \sqrt{\zeta^2 - 1}) = w^2 (\zeta^2 - \zeta^2 + 1) = w^2$$

Taking Laplace transform

$$y(t) = \frac{w^2 k_p B}{w^2} \left[1 - \frac{w(\zeta + \sqrt{\zeta^2 - 1})e^{-w(\zeta - \sqrt{\zeta^2 - 1})t} - w(\zeta - \sqrt{\zeta^2 - 1})e^{-w(\zeta + \sqrt{\zeta^2 - 1})t}}{w(\zeta + \sqrt{\zeta^2 - 1}) - w(\zeta - \sqrt{\zeta^2 - 1})} \right]$$

$$y(t) = k_p B \left[1 - \frac{(\zeta + \sqrt{\zeta^2 - 1})e^{-w(\zeta - \sqrt{\zeta^2 - 1})t} - (\zeta - \sqrt{\zeta^2 - 1})e^{-w(\zeta + \sqrt{\zeta^2 - 1})t}}{2\sqrt{\zeta^2 - 1}} \right]$$

$$y(t)$$

$$= k_p B \left[1 - \frac{\zeta e^{-w\zeta t} (e^{w(\sqrt{\zeta^2 - 1})t} - e^{-w(\sqrt{\zeta^2 - 1})t}) - \sqrt{\zeta^2 - 1} e^{-w\zeta t} (e^{w(\sqrt{\zeta^2 - 1})t} + e^{-w(\sqrt{\zeta^2 - 1})t})}{2\sqrt{\zeta^2 - 1}} \right]$$

$$y(t) = k_p B \left[1 - \frac{e^{-w\zeta t} \{ 2\zeta \sinh w(\sqrt{\zeta^2 - 1})t + 2\sqrt{\zeta^2 - 1} \cosh w(\sqrt{\zeta^2 - 1})t \}}{2\sqrt{\zeta^2 - 1}} \right]$$

$$y(t) = k_p B \left[1 - e^{-w\zeta t} \left\{ \frac{\zeta}{\sqrt{\zeta^2 - 1}} \sinh w(\sqrt{\zeta^2 - 1})t + \cosh w(\sqrt{\zeta^2 - 1})t \right\} \right]$$

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Note: $\sinh a = \frac{e^a - e^{-a}}{2}$, $\cosh a = \frac{e^a + e^{-a}}{2}$

Case 2: $\zeta = 1$, Critically damped system, roots are **real and equal**

$$\bar{y}(s) = \frac{k_p/\tau^2}{s^2 + 2\zeta s/\tau + 1/\tau^2} \frac{B}{s}$$

$$\bar{y}(s) = \frac{w^2 k_p}{(s + w)^2} \frac{B}{s}$$

$$y(t) = \frac{w^2 k_p B}{w^2} (1 - e^{-wt} - we^{-wt})$$

$$y(t) = k_p B (1 - e^{-wt} - we^{-wt})$$

Case 3: $\zeta < 1$, under damped system, roots are **complex and negative**

$$\bar{y}(s) = \frac{k_p/\tau^2}{(s - p)(s - q)} \frac{B}{s}$$

$$p = -\frac{\zeta}{\tau} + i\frac{\sqrt{1 - \zeta^2}}{\tau}, \quad q = -\frac{\zeta}{\tau} - i\frac{\sqrt{1 - \zeta^2}}{\tau}$$

Let, $\frac{1}{\tau} = w$, $a = -p$, $b = -q$

$$a = w\zeta - wi\sqrt{1 - \zeta^2}, \quad b = w\zeta + wi\sqrt{1 - \zeta^2}$$

Therefore,

$$\bar{y}(s) = \frac{w^2 k_p}{(s + a)(s + b)} \frac{B}{s}$$

Note: Laplace transform of $\frac{K}{s(s+a)(s+b)} = \frac{K}{ab} \left[1 - \frac{be^{-at} - ae^{-bt}}{b-a} \right]$

$$a \cdot b = w^2 (\zeta - i\sqrt{1 - \zeta^2}) \cdot (\zeta + i\sqrt{1 - \zeta^2}) = w^2 (\zeta^2 - (-1)(1 - \zeta^2)) = w^2$$

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$$b - a = w(\zeta + i\sqrt{1 - \zeta^2}) - w(\zeta - i\sqrt{1 - \zeta^2}) = 2wi\sqrt{1 - \zeta^2}$$

Taking Laplace transform

$$y(t) = \frac{w^2 k_p B}{w^2} \left[1 - \frac{w(\zeta + i\sqrt{1 - \zeta^2})e^{-w(\zeta - i\sqrt{1 - \zeta^2})t} - w(\zeta - i\sqrt{1 - \zeta^2})e^{-w(\zeta + i\sqrt{1 - \zeta^2})t}}{w(\zeta + i\sqrt{1 - \zeta^2}) - w(\zeta - i\sqrt{1 - \zeta^2})} \right]$$

$$y(t) = k_p B \left[1 - \frac{\zeta e^{-w\zeta t} (e^{iw(\sqrt{1 - \zeta^2})t} - e^{-iw(\sqrt{1 - \zeta^2})t}) + i\sqrt{1 - \zeta^2} e^{-w\zeta t} (e^{iw(\sqrt{1 - \zeta^2})t} + e^{-iw(\sqrt{1 - \zeta^2})t})}{2i\sqrt{1 - \zeta^2}} \right]$$

$$y(t) = k_p B \left[1 - \frac{e^{-w\zeta t} \{ (2i)\zeta \sin w(\sqrt{1 - \zeta^2})t + i(2)\sqrt{1 - \zeta^2} \cos w(\sqrt{1 - \zeta^2})t \}}{2i\sqrt{1 - \zeta^2}} \right]$$

$$y(t) = k_p B \left[1 - e^{-w\zeta t} \left\{ \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin w(\sqrt{1 - \zeta^2})t + \cos w(\sqrt{1 - \zeta^2})t \right\} \right]$$

Note: $\sin a = \frac{e^{ia} - e^{-ia}}{2i}, \quad \cos a = \frac{e^{ia} + e^{-ia}}{2}$

$$p \cos A + q \sin A = r \sin(A + \theta), \text{ where, } r = \sqrt{p^2 + q^2}, \quad \theta = \tan^{-1} \frac{p}{q}$$

Here, $p = \frac{\zeta}{\sqrt{1 - \zeta^2}}, q = 1$, then $r = \sqrt{p^2 + q^2} = \frac{1}{\sqrt{1 - \zeta^2}}, \theta = \tan^{-1} \frac{p}{q} = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}$

Therefore,

$$y(t) = k_p B \left[1 - e^{-w\zeta t} \frac{1}{\sqrt{1 - \zeta^2}} \sin \left\{ w(\sqrt{1 - \zeta^2})t + \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \right\} \right]$$

$$y(t) = k_p B \left[1 - e^{-w\zeta t} \frac{1}{\sqrt{1 - \zeta^2}} \sin \left\{ \left(\frac{\sqrt{1 - \zeta^2}}{\zeta} \right) t + \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta} \right\} \right]$$

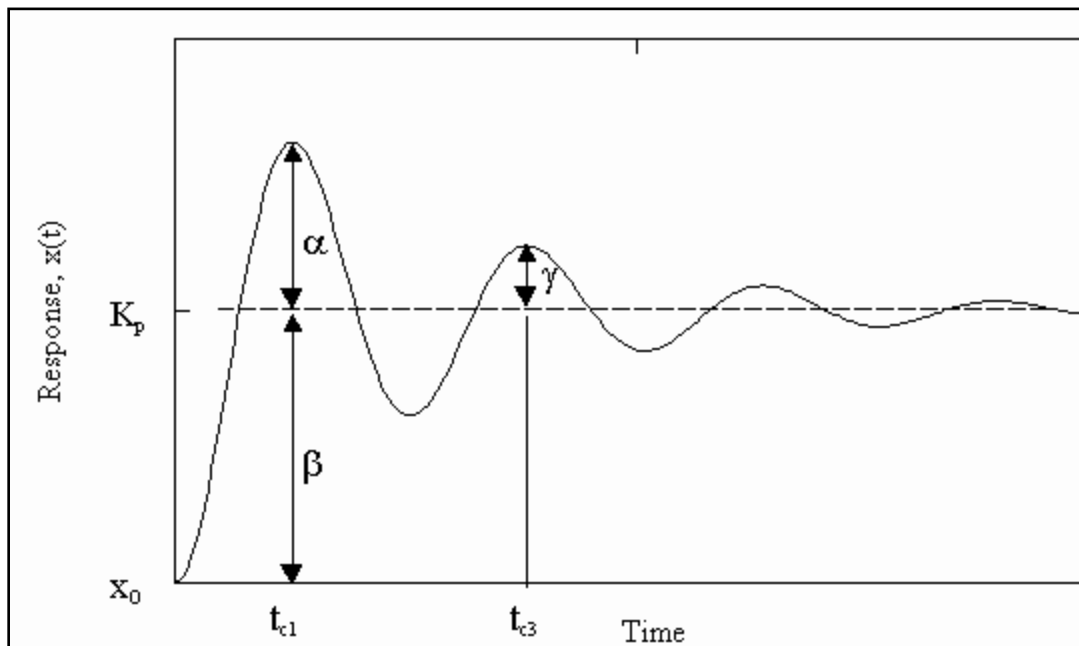
$$\text{Angular Frequency} = \frac{\sqrt{1 - \zeta^2}}{\zeta} \text{ and Phase Lag} = \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{\zeta}$$

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Response of second order system with impulse and sinusoidal input:

See Book Coughnaowr Page 98, 99 and 100

Response analysis of second order system when system experienced step input (Under damped system):



Overshoot: How much response exceeds the ultimate value (α/β)

$$Overshoot (\alpha/\beta) = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

Decay Ratio: ratio of the two successive peaks ()

$$Decay Ratio \left(\frac{\gamma}{\alpha}\right) = \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right) = (Overshoot)^2$$

Rise time: Time required for the response to first reach its ultimate value.

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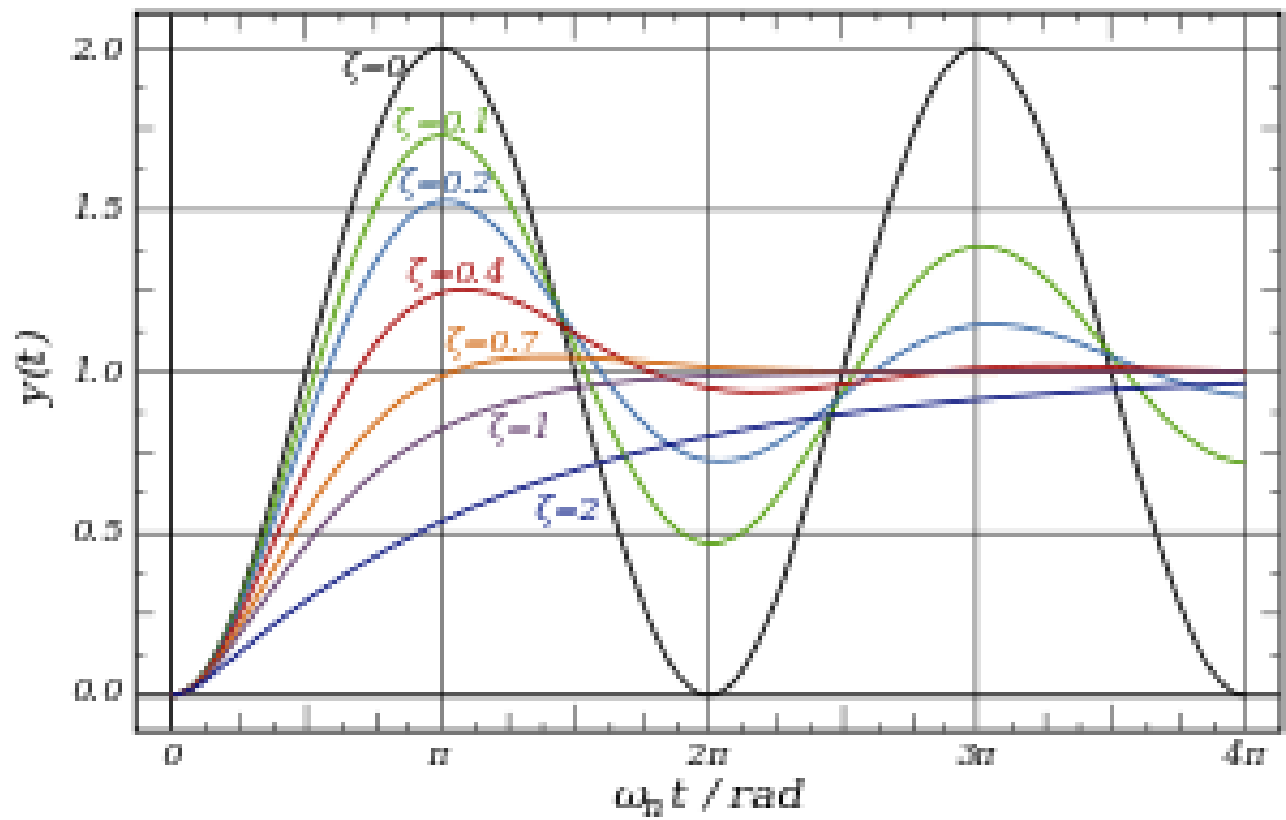
$$\text{Angular Frequency } (w) = \frac{\sqrt{\zeta^2 - 1}}{\zeta}$$

$$\text{Cyclic Frequency } (f) = \frac{w}{2\pi} = \frac{\sqrt{\zeta^2 - 1}}{2\pi\zeta}$$

$$\text{Period of Oscillation } (T) = \frac{1}{f} = \frac{2\pi\zeta}{\sqrt{\zeta^2 - 1}}$$

Natural Period of Oscillation : Period of Oscillation at $\zeta = 0$

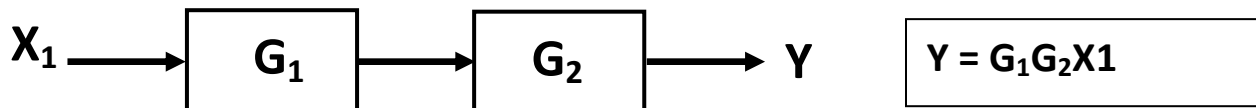
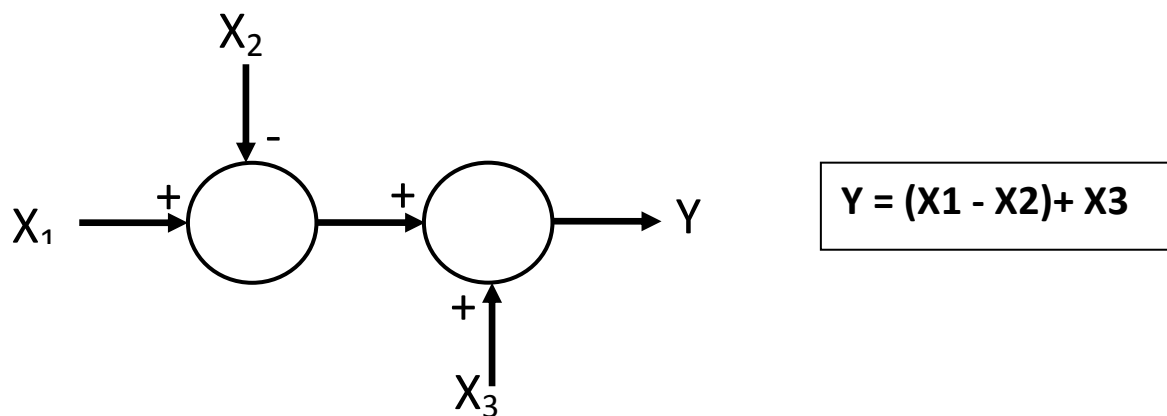
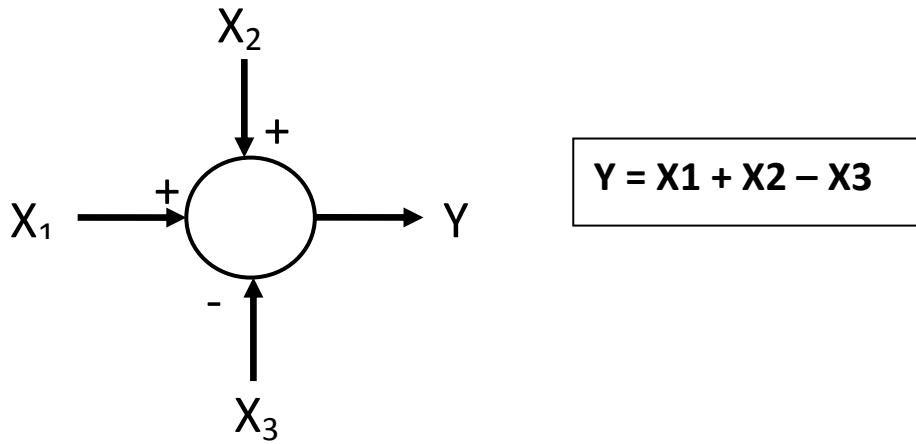
Response analysis of second order system when system experienced step input:

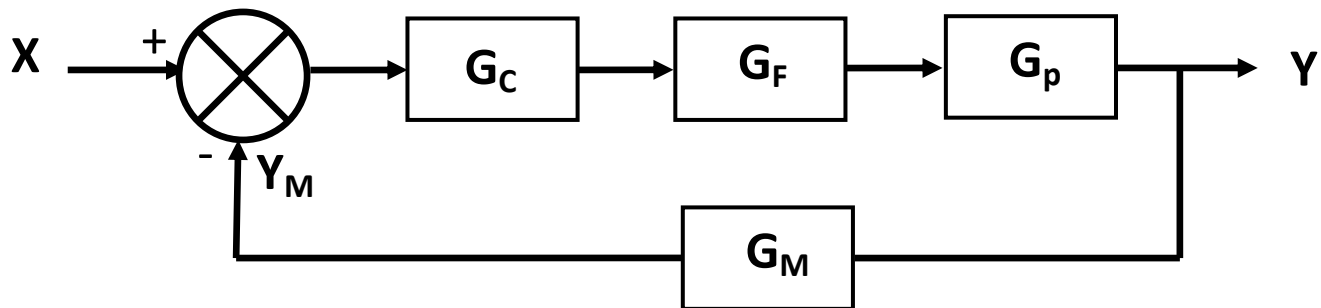
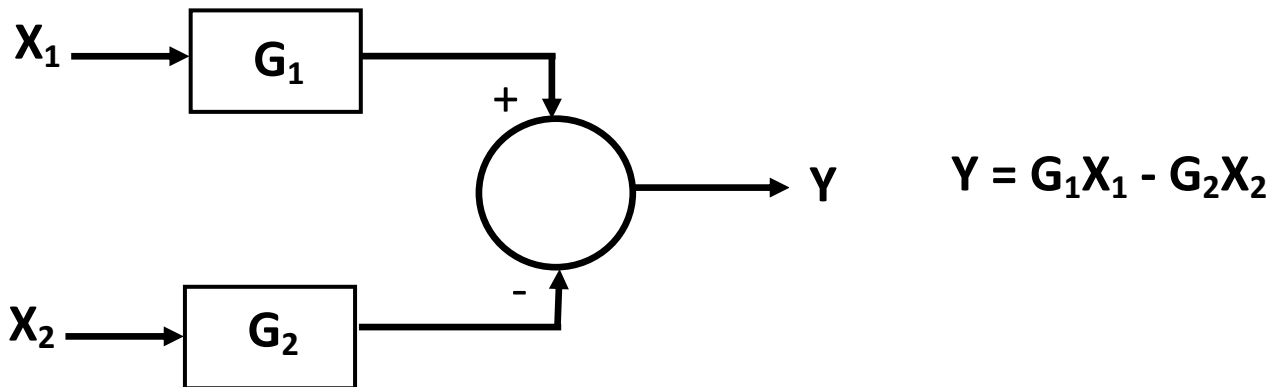


Transportation Lag

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Block Diagram:





Develop Transfer function of the above block diagram:

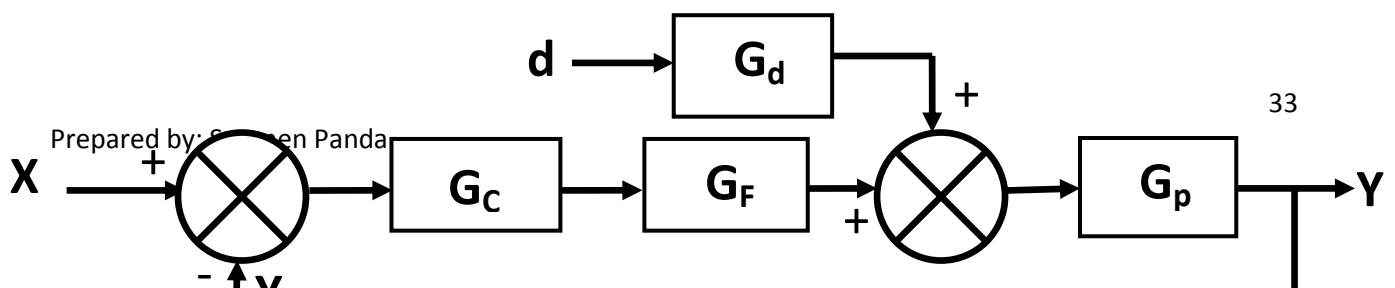
$$Y(s) = G_C G_F G_P (X - Y_M)$$

$$Y(s) = G_C G_F G_P (X - G_M Y)$$

$$Y(s)(G_C G_F G_P G_M + 1) = G_C G_F G_P X$$

$$\frac{Y(s)}{X(s)} = \frac{G_C G_F G_P}{(G_C G_F G_P G_M + 1)} = G(s)$$

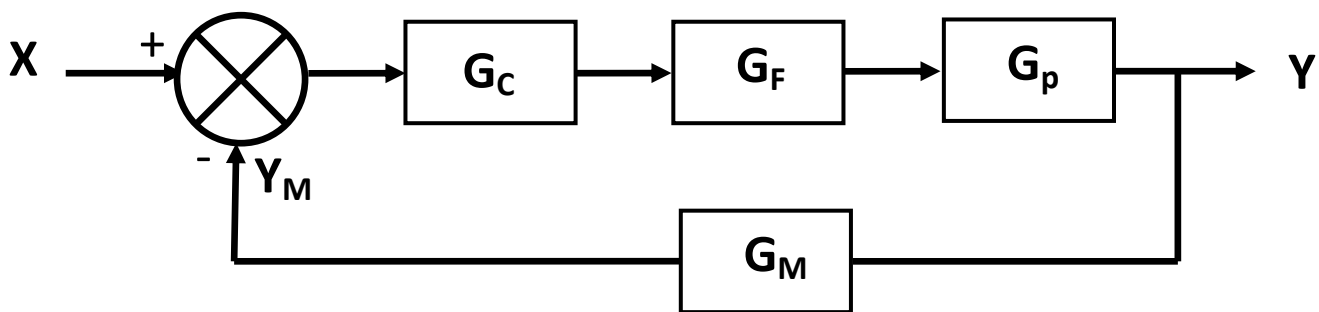
$(G_C G_F G_P G_M + 1) = 0$, is called Characteristic equation



Develop transfer function for the above block diagram:

Determination of $Y(s)$ is difficult in direct method, so we determine the $Y(s)$ value once omitting $d(s)$ and then omitting $X(s)$

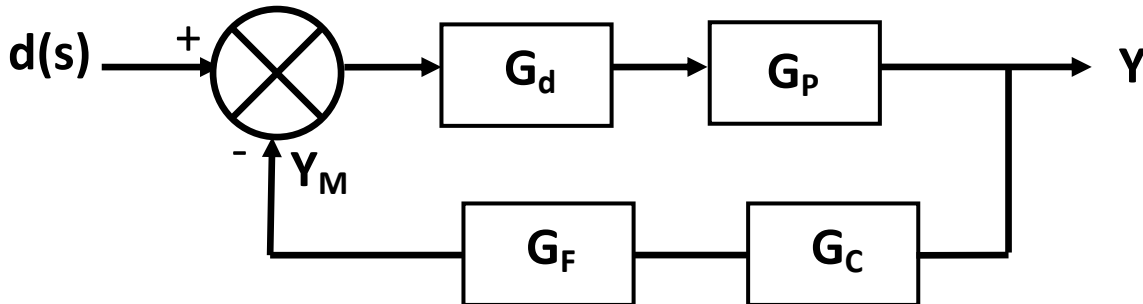
Omitting $d(s)$: i.e. $d(s) = 0$



Develop Transfer function of the block diagram:

$$Y(s) = \frac{G_C G_F G_P}{(G_C G_F G_P G_M + 1)} X(s)$$

Omitting $X(s)$: i.e. $X(s) = 0$



Develop Transfer function of the block diagram:

$$Y(s) = \frac{G_d G_P}{(G_C G_F G_P G_M + 1)} d(s)$$

When, $X(s)$ and $d(s)$ are not equal to zero,

$$Y(s) = \frac{G_C G_F G_P}{(G_C G_F G_P G_M + 1)} X(s) + \frac{G_d G_P}{(G_C G_F G_P G_M + 1)} d(s)$$

When $d(s) = 0$ (No disturbance in the system)

$$Y(s) = \frac{G_C G_F G_P}{(G_C G_F G_P G_M + 1)} X(s)$$

This type of problem called regulatory problem Regulatory problem

When $X(s) = 0$ (No change in set point in the system)

$$Y(s) = \frac{G_d G_P}{(G_C G_F G_P G_M + 1)} d(s)$$

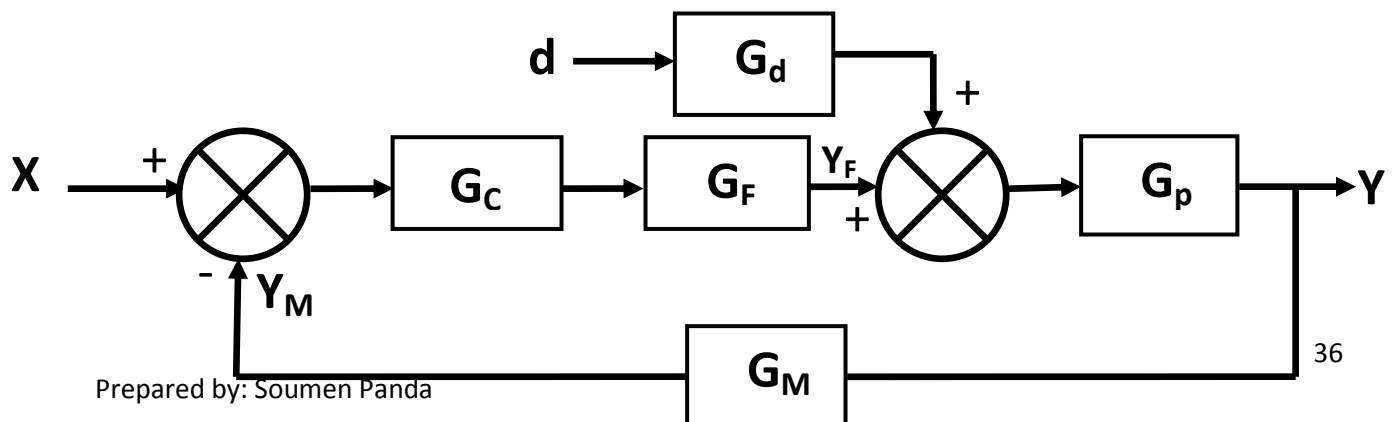
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This type of problem called servo problem Regulatory problem

Transient response:

Variation of system response with time before reaching steady state is called transient response.

Aim: To find variation of desired output of a system with time when system is experienced an external disturbance or a change in set point.



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Response of above system in Laplace domain can be expressed as following

$$Y(s) = \frac{G_C G_F G_P}{(G_C G_F G_P G_M + 1)} X(s) + \frac{G_d G_P}{(G_C G_F G_P G_M + 1)} d(s)$$

Response of the system can be found using following steps

Step 1. Knowing values of G_C , G_F , G_P , G_M , G_d , X and d we can find $Y(s)$ servo or regulatory problem.

Step 2. Take inverse Laplace of $Y(s)$ to get $Y(t)$, response in time domain. Find initial and final value of response with the substitution of $t \rightarrow 0$ and $t \rightarrow \infty$, respectively in the expression of $Y(t)$.

Alternate Step 2. Find initial and final value directly from $Y(s)$ with the application of initial and final value theorem.

Example:

$$G_C = K; G_M = 1; G_F = 1; G_d = 1/A; G_P = A/(\tau s + 1)$$

Case 1:

Let us assume $X(s)$ $1/s$ (step response) and $d = 0$ (regulatory problem)

Therefore,

$$Y(s) = \frac{G_C G_F G_P}{(G_C G_F G_P G_M + 1)} X(s) = \frac{(K)(1) \left(\frac{A}{(\tau s + 1)} \right)}{\left[K(1) \left(\frac{A}{(\tau s + 1)} \right) (1) + 1 \right]} \frac{1}{s} = \frac{KA}{[\tau s + 1 + KA]} \frac{1}{s}$$

Final value:

Apply final value theorem

$$Y(t \rightarrow \infty) = sY(s)(s \rightarrow 0)$$

$$Y(t \rightarrow \infty) = s \frac{KA}{[\tau s + 1 + KA]} \frac{1}{s} \quad (s \rightarrow 0)$$

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$$Y(t \rightarrow \infty) = \frac{KA}{1 + KA}$$

Offset = value of set point in time domain – Value of response $Y(t \rightarrow \infty)$

$$\text{Offset} = 1 - \frac{KA}{1+KA} = \frac{1}{1+KA}$$

Offset is defined as the amount of deviation of ultimate response from the expected value. Here, expected ultimate response of the system is 1 and ultimate response of the system is $\frac{KA}{1+KA}$.

Initial value:

Apply initial value theorem

$$Y(t \rightarrow 0) = sY(s) \quad (s \rightarrow \infty)$$

$$Y(t \rightarrow 0) = s \frac{KA}{[\tau s + 1 + KA]} \frac{1}{s} \quad (s \rightarrow \infty)$$

$$Y(t \rightarrow 0) = \frac{KA/s}{\tau + 1/s + KA/s} = 0$$

Case 2:

Let us assume $d(s) = 1/s$ (step response) and $X(s) = 0$ (Servo problem)

$$Y(s) = \frac{G_d G_p}{(G_c G_f G_p G_m + 1)} d(s) = \frac{(1/A) \left(\frac{A}{(\tau s + 1)} \right)}{\left[K(1) \left(\frac{A}{(\tau s + 1)} \right) (1) + 1 \right]} \frac{1}{s} = \frac{1}{[\tau s + 1 + KA]} \frac{1}{s}$$

Final value:

Apply final value theorem

$$Y(t \rightarrow \infty) = sY(s) \quad (s \rightarrow 0)$$

$$Y(t \rightarrow \infty) = s \frac{1}{[\tau s + 1 + KA]} \frac{1}{s} \quad (s \rightarrow 0)$$

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$$Y(t \rightarrow \infty) = \frac{1}{1 + KA}$$

Offset = 0 – Value of response $Y(t \rightarrow \infty)$

$$\text{Offset} = 0 - \frac{1}{1+KA} = -\frac{1}{1+KA}$$

Offset is defined as the amount of deviation of ultimate response from the expected value. Here, expected ultimate response of the system is 0 and ultimate response of the system is $\frac{-1}{1+KA}$.

Initial value:

Apply initial value theorem

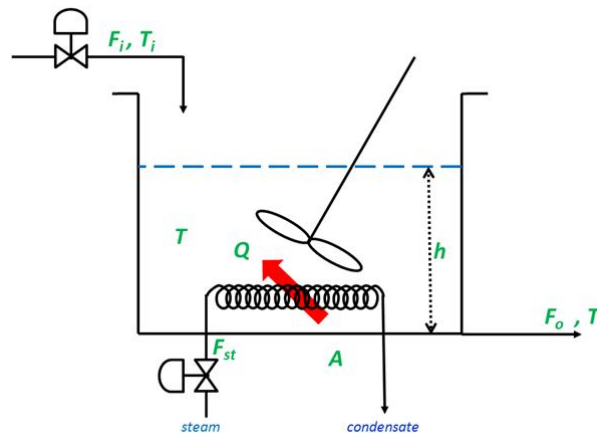
$$Y(t \rightarrow 0) = sY(s) \quad (s \rightarrow \infty)$$

$$Y(t \rightarrow 0) = s \frac{1}{[\tau s + 1 + KA]} \frac{1}{s} \quad (s \rightarrow \infty)$$

$$Y(t \rightarrow 0) = \frac{1/s}{\tau + 1/s + KA/s} = 0$$

Transient response of a real system:

Stirred tank heater:



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Apply energy balance:

At any time t,
$$Q + F\rho C(T_i - T) = \rho CV \frac{dT}{dt}$$

At steady state,
$$Q_s + F\rho C(T_{is} - T_s) = \rho CV \frac{dT_s}{dt}$$

In deviation form

$$Q' + F\rho C(T_i' - T') = \rho CV \frac{dT'}{dt}$$

Where, $Q' = Q - Q_s$ and $T_i' = T_i - T_{is}$ and $T' = T - T_s$

Apply Laplace transform in both side and substitute $T' = 0$, we get

$$\bar{T}(s) \left(\frac{V}{F} s + 1 \right) = \frac{\bar{Q}(s)}{F\rho C} + \bar{T}_i$$

$$\bar{T}(s) = \frac{\frac{1}{F\rho C}}{(\tau s + 1)} \bar{Q}(s) + \frac{1}{(\tau s + 1)} \bar{T}_i(s)$$

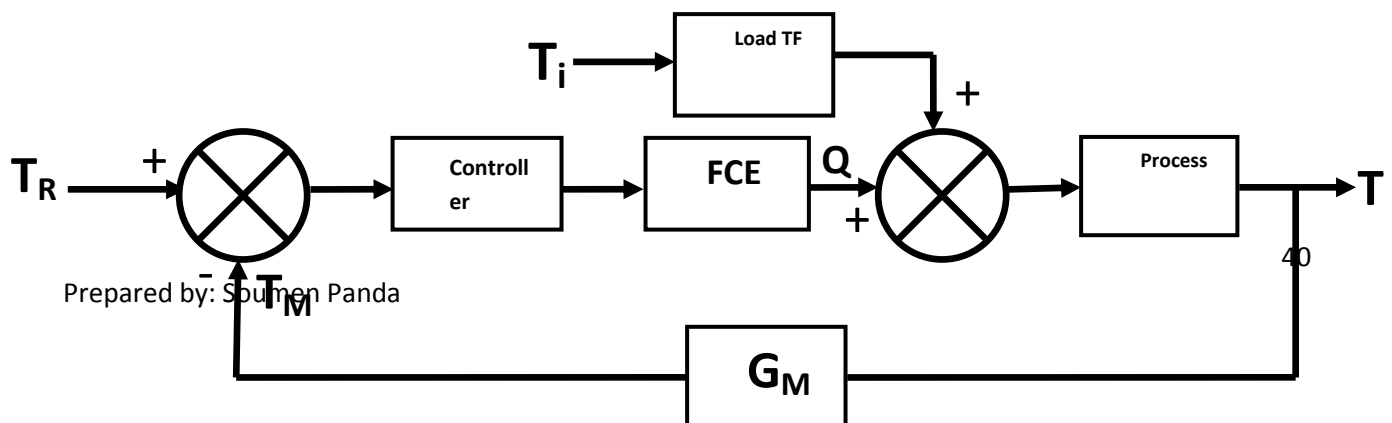
Where, $\tau = \frac{V}{F}$

We can write above expression as

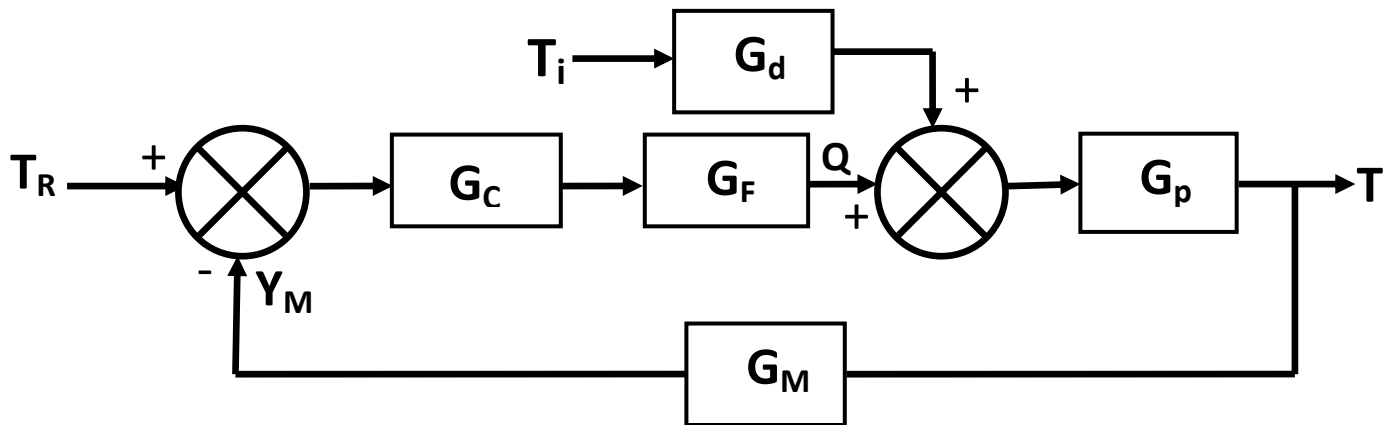
$$\bar{T}(s) = G_p \bar{Q}(s) + G_d G_p \bar{T}_i(s) \quad (\text{open loop})$$

Where, $G_p = \frac{1/(F\rho C)}{(\tau s + 1)}$ and $G_d = F\rho C$

Response in closed loop system:



We can write the above system as following



$$\text{Here, } Y(s) = G_P Y_F(s) + G_d G_P d(s)$$

$$T(s) = \frac{G_C G_F G_P}{(G_C G_F G_P G_M + 1)} T_R(s) + \frac{G_d G_P}{(G_C G_F G_P G_M + 1)} T_i(s)$$

Ultimate response can be found for servo and regulatory problem for different value of G_C

$$G_C = K \text{ — proportional controller}$$

$$G_C = K \left(1 + \frac{1}{\tau_I s} \right) \text{ — proportional integral controller}$$

$$G_C = K \left(1 + \frac{1}{\tau_I s} + \tau_D s \right) \text{ — proportional integral derivative controller}$$

$$\text{Here, } G_P = \frac{1/(F\rho C)}{(\tau s + 1)} \text{ and } G_d = F\rho C$$

$$\text{Assume, } G_M = 1, G_F = 1$$

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Servo Problem:

No change in disturbance ($T_i' = 0$)

Response for Proportional – Integral controller (PI)

Therefore,

$$Y(s) = \frac{G_C G_F G_P}{(G_C G_F G_P G_M + 1)} X(s) = \frac{\left[K \left(1 + \frac{1}{\tau_I s} \right) \right] (1) \left(\frac{A}{(\tau s + 1)} \right)}{\left[\left[K \left(1 + \frac{1}{\tau_I s} \right) \right] (1) \left(\frac{A}{(\tau s + 1)} \right) (1) + 1 \right]} \frac{1}{s}$$

$$= \frac{KA(\tau_I s + 1)}{[\tau_I s(\tau s + 1) + KA(\tau_I s + 1)]} \frac{1}{s}$$

Final value:

Apply final value theorem

$$Y(t \rightarrow \infty) = sY(s)(s \rightarrow 0)$$

$$Y(t \rightarrow \infty) = s \frac{KA(\tau_I s + 1)}{[\tau_I s(\tau s + 1) + KA(\tau_I s + 1)]} \frac{1}{s} \quad (s \rightarrow 0)$$

$$Y(t \rightarrow \infty) = \frac{KA}{KA} = 1$$

Offset = value of set point in time domain – Value of response $Y(t \rightarrow \infty)$

Offset = $1 - 1 = 0$

Offset is defined as the amount of deviation of ultimate response from the expected value. Here, expected ultimate response of the system is 1 and ultimate response of the system is $\frac{KA}{1+KA}$.

Regulatory problem:

No change in point of vessel temperature ($T_R' = 0$)

Response for PI controller

$$(s) = \frac{G_d G_P}{(G_C G_F G_P G_M + 1)} d(s) = \frac{(1/A) \left(\frac{A}{(\tau s + 1)} \right)}{\left[K \left(1 + \frac{1}{\tau_I s} \right) (1) \left(\frac{A}{(\tau s + 1)} \right) (1) + 1 \right]} \frac{1}{s} = \frac{\tau_I s}{[\tau_I s(\tau s + 1) + KA]} \frac{1}{s}$$

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Final value:

Apply final value theorem

$$Y(t \rightarrow \infty) = sY(s) \quad (s \rightarrow 0)$$

$$Y(t \rightarrow \infty) = s \frac{\tau_I s}{[\tau_I s(\tau s + 1) + KA]} \frac{1}{s} \quad (s \rightarrow 0)$$

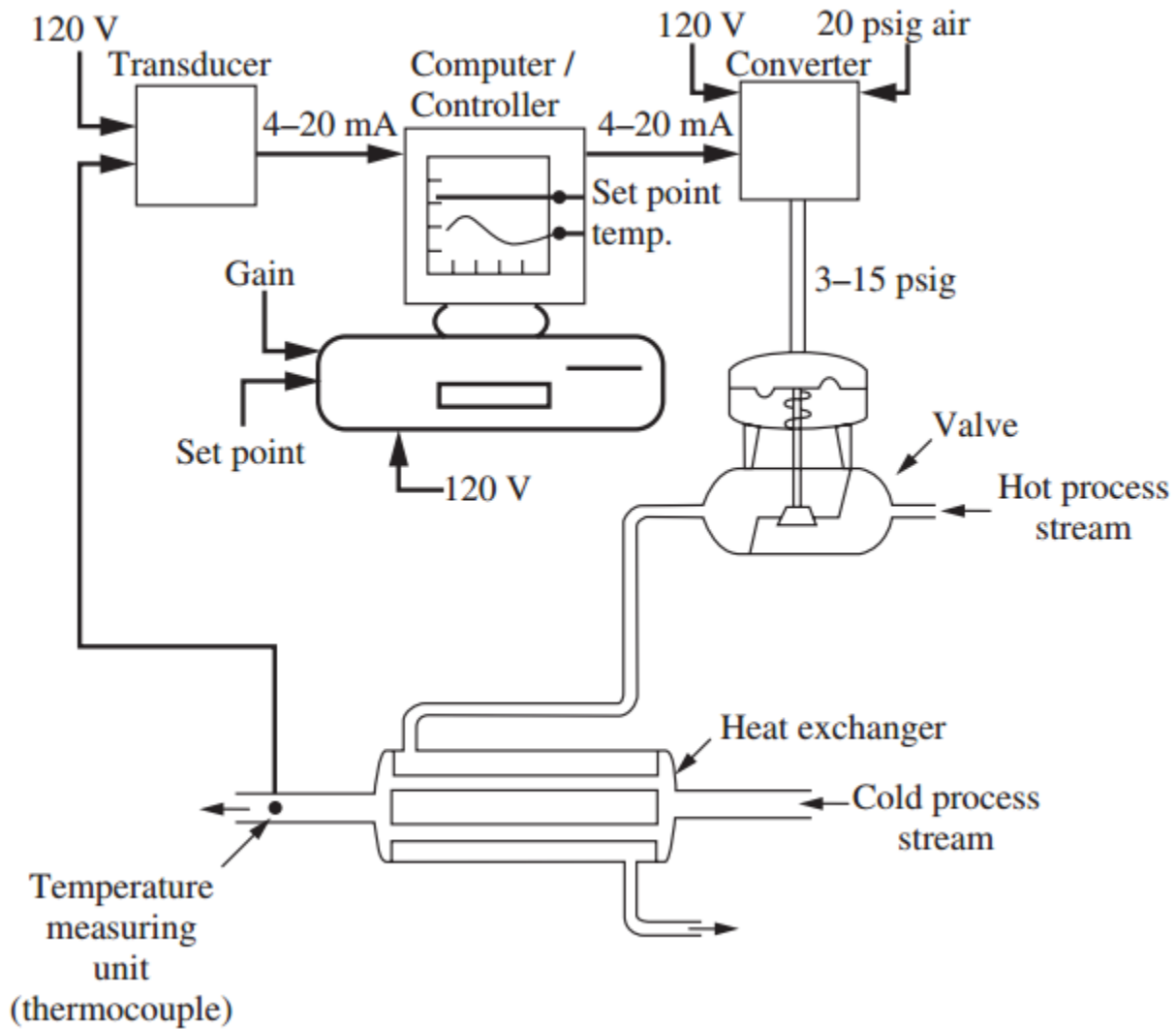
$$Y(t \rightarrow \infty) = 0$$

Offset = 0 – Value of response $Y(t \rightarrow \infty)$

$$\text{Offset} = 0 - 0 = 0$$

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Controller and Final control element



Transducer: Takes Input as temperature and provide output as current

Controller recorder: Takes Input as current and provide output as current

Converter: Takes Input as current and provide output pressure

Control valve: Takes Input as air pressure and provide output as flow rate

Note: In the control system an external power of 120 V is needed to run each component like transducer, controller and converter. A source of 20 psig is needed for the converter.

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Controller: Controller may be thought of as a device that receives the error signal and provides corrective instruction to the final control element as output of it.

Final Control Element: The final control element may be regarded as a device that produces corrective action on the process. The corrective action is regarded as mathematically related to the output signal from the controller.

Types of Controller:

1. **Proportional Controller**
2. **Proportional – Integral Controller**
3. **Proportional – Integral – Derivative Controller**

Proportional Controller: The proportional controller produces an output signal that is proportional to the error (ϵ).

This action may be expressed as

$$P = K\epsilon + P_s$$

P = Output signal from controller, psig or mA

K = Gain or sensitivity

ϵ = error = set point – measured variable

P_s = Output signal of the controller when error is zero

P_s can be adjusted to obtain the required output signal when the control system is at its steady state and error is zero.

At steady state $P = P_s$

Therefore,

$$P' = P - P_s = K\epsilon$$

Taking Laplace transform

$$\bar{P}(s) = K\bar{\epsilon}(s)$$

$$\frac{\bar{P}(s)}{\bar{\epsilon}(s)} = K, \text{ Where } K \text{ is called proportional gain}$$

In proportional controller instead of Gain proportional Band is used

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Proportional Band (Pb): The error required to move valve from fully close to fully open. This is expressed in percentage.

Let a controller can control range of a parameter is **R**, but the parameter can vary only **P** unit when the valve move from fully close to fully open.

Therefore, *Proportional Band* = $\frac{P}{R} \times 100 \%$

Example:

A pneumatic proportional controller is used to control temperature within the range of 60 to 100 °C. The controller is adjusted so that the output pressure goes from 3 psi to 15 psi (Valve move fully close to fully open) as the measured temperature goes from 71 to 75 °C with the set point held constant. Find the Gain and Proportional Band.

$$\text{Proportional Band} = \frac{75 - 71}{100 - 60} \times 100\% = 10\%$$

$$\text{Proportional Gain} = \frac{15 - 3}{75 - 71} = 3 \text{ psig}/^{\circ}\text{C}$$

Now assume that the proportional band of the controller is changed to 75%. Find gain and temperature change necessary to cause a valve to go fully open to close.

As Pb is 75%, so temperature can vary up to $(100-60) \times 0.75 = 30$ °C when valve go fully open to fully close.

$$\text{Proportional Gain} = \frac{15 - 3}{30} = 0.4 \text{ psig}/^{\circ}\text{C}$$

Therefore, $\text{Proportional Gain} \propto \frac{1}{\text{Proportional Band}}$

ON – OFF Controller: This is a special type of proportional controller.

If Gain K of the proportional controller is very high, the valve will move from one extreme position to other when controller experienced very small error. This very sensitive action is called On – OFF action, because the valve is either open or close i.e valve acts like a switch.

The band width of an ON – OFF controller is approximately zero.

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Proportional – Integral control:

This mode of controller is described by the relationship

$$P = K\epsilon + \frac{K}{\tau_I} \int_0^t \epsilon dt + P_s$$

Where, $\tau_I = \text{Integral time, min}$

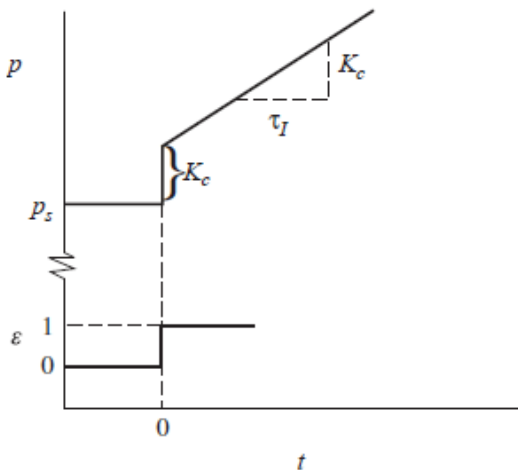
To visualize the response of this controller, consider the response to a unit step change in error.

$$\epsilon = 1$$

Therefore,

$$P(t) = K + \frac{K}{\tau_I} t + P_s$$

P changes suddenly by K amount and then changes linearly with time at a rate of $\frac{K}{\tau_I}$



Apply Laplace transform to get Transfer function

$$\frac{\bar{P}(s)}{\bar{\epsilon}(s)} = K \left(1 + \frac{1}{\tau_I s} \right)$$

$\frac{1}{\tau_I}$ is called reset time

Proportional – Derivative Controller:

This mode of controller is described by the relationship

$$P = K\epsilon + K\tau_D \frac{d\epsilon}{dt} + P_s$$

Where, $\tau_D = \text{Derivative time, min}$

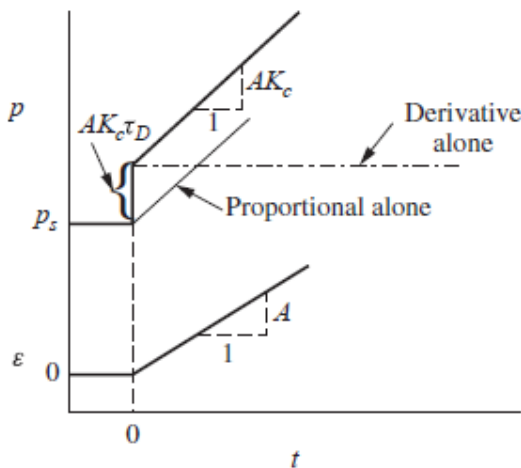
To visualize the response of this controller, consider the response to a linear change in error.

$$\epsilon = At$$

Therefore,

$$P(t) = AKt + AK\tau_D + P_s$$

The effect of derivative action in this case is to anticipate the linear change in error by adding additional output $AK\tau_D$ to the proportional action.



Apply Laplace transform to get Transfer function

$$\frac{\bar{P}(s)}{\bar{\epsilon}(s)} = K(1 + \tau_D s)$$

Proportional – Integral – Derivative Controller:

This mode of controller is described by the relationship

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$$P = K\epsilon + \frac{K}{\tau_I} \int_0^t \epsilon dt + K\tau_D \frac{d\epsilon}{dt} + P_s$$

Similarly we get Transfer function

$$\frac{\bar{P}(s)}{\bar{\epsilon}(s)} = K \left(1 + \tau_D s + \frac{1}{\tau_I s} \right)$$

Motivation of addition of Integral and Derivative control mode:

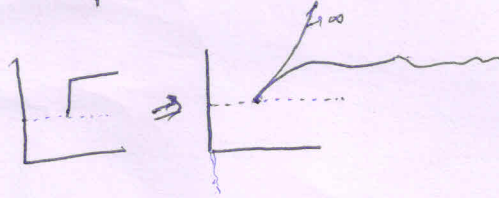
Use of only proportional controller we get non zero offset that is not expected. PI controller has the ability to make offset zero. Though PI controller provides zero offset, it produces an oscillatory behavior.

Use of PID controller arrests rise of control variable more quickly and return rapidly to the original value with little or no oscillation.

Note: If some offset is tolerable proportional action would likely be selected. If no offset were tolerable, integral action would be added. If excessive oscillation needs to be eliminated, derivative action might be added.

Stability

For a bounded input (step, pulse, sinusoidal) if a system produces bounded output. Then the system is called stable system.



stability Conditions

$$\frac{Y(s)}{F(s)} = G(s) = \frac{N(s)}{D(s)}$$

$$Y(s) = \frac{G_{OL}}{1+G_{OL}} X(s) + \frac{G_{OL}}{1+G_{OL}} d(s)$$

$$Y(s) = \frac{N_1(s)}{D_1(s)} X(s) + \frac{N_2(s)}{D_1(s)} d(s)$$

$$Y(s) = \frac{N_1(s)}{(s-p_1)(s-p_2)(s-p_3)^m(s-p_4)(s-p_4^*)} \cdot \frac{A}{s} + \dots$$

Take inverse of laplace.

$$Y(s) = \frac{C_5}{s} + \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \left\{ \frac{C_{31}}{s-p_3} + \frac{C_{32}}{(s-p_3)^2} + \dots \right\} + \frac{C_4}{s-p_4} + \frac{C_4^*}{s-p_4^*}$$

$$Y(t) = C_5 + C_1 e^{p_1 t} + C_2 e^{p_2 t} + C_{31} e^{p_3 t} + C_{32} \frac{t}{1!} e^{p_3 t} + \dots + C_4 e^{p_4 t} + C_4^* e^{p_4^* t}$$

When $t \rightarrow \infty$, if p_1, p_2, p_3, p_4 & p_4^* are positive (or any value of p_1) they $Y(t) \rightarrow \infty$ (unbound)

if p_1, p_2, p_3, p_4 & p_4^* are negative the $Y(t) \rightarrow \text{finite}$. (bound)

So, roots of the characteristic equation must be negative.

~~Zero~~ $1 + G_{OL} = 0 \Rightarrow$ find roots. (Poles)

Routh Hurwitz Criterion

Express characteristic equation in terms of polynomial equation.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0.$$

Test 1:

If all $\{a_i\}$ $i=1$ to n are +ve go for test 2.
Otherwise system is unstable.

Test 2: Construct Routh-Hurwitz array

	column 1	column 2	column 3
R_1	a_0	a_2	a_4
R_2	a_1	a_3	a_5
R_3	A_1	A_2	A_3
R_4	B_1	B_2	B_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
R_N	N_1	N_2	N_3

$$A_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$A_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$A_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$B_1 = \frac{A_1 a_3 - a_1 A_2}{A_1}$$

* If any element in the first column is -ve then the system is unstable.

* The number of sign change in the first column is same as number of +ve roots of the characteristic equation.

Limitation: 1. It is valid only if the characteristic equation is algebraic.

2. If any coefficient of the characteristic equation is complex or contain power of 'exp' this criterion cannot be applied.

3. System contains transportation lag can not be applied.

4. It gives information about absolute stability of system not degree of stability.

5. It gives information about how many roots are on RHS of s-plane. values of the roots are not available. Also can not distinguish between real & complex roots.

Root locus

In control theory and stability theory, root locus analysis is a graphical method for examining how the roots of a system change with variation of a certain system parameter, commonly a gain within a feedback system.

1) Root locus is symmetrical about real axis, roots are real or complex conjugate

2) As K ~~is~~ increases from $0 \rightarrow \infty$, each branch of root locus originates from open loop poles as $K=0$ and terminate^{on} either open loop zero or on ∞ .

$$G(s)H(s) = K \frac{N(s)}{D(s)}$$

Characteristic eqn

$$1 + G(s)H(s) = 0$$

$$\Rightarrow 1 + K \frac{N(s)}{D(s)} = 0$$

$$\Rightarrow D(s) + K N(s) = 0$$

$$K=0, \\ D(s)=0$$

Poles ~~zeros~~ are the roots of characteristic eqn

$$K \rightarrow \infty \\ K N(s) = 0 \text{ [as } D(s) \ll K N(s)]$$

$$N(s) = 0$$

~~Poles~~ Zeros are the roots of characteristic eqn

- 3) A point of the real axis lies on the root locus if the number of open loop poles + zeros on real axis to the right of the point is odd.

$$1 + G(s)H(s) = 0$$

angle $\angle G(s)H(s) = \pm (2i+1)180^\circ$

$$\Rightarrow G(s)H(s) = -1$$

$$\Rightarrow |G(s)H(s)| = 1 \quad \text{magnitude}$$

4) Center of gravity = $\frac{(\text{sum of poles}) - (\text{sum of zeros})}{(\# \text{ poles}) - (\# \text{ of zeros})}$

Angle of asymptotes $\theta = \frac{(2i+1)}{n-m} \times 180^\circ$ $i = 0, 1, 2, 3, \dots, \frac{n-m}{2}-1$
total $(n-m)$ numbers

- 5) Break away point:

The point at which two root loci emerging from adjacent poles on the real axis, intersect and then leave the real axis is called break away point.

Method 1: $\sum_{i=1}^m \frac{1}{s-z_i} = \sum_{j=1}^n \frac{1}{s-p_j}$ roots of the equation gives break away point.

Method 2: Find characteristic eqn

$$1 + G_{OL}(s) = 0$$

Express the equation in terms of

$$K = (\quad)$$

$$\text{Find } \frac{dK}{ds} = (\quad) = 0$$

Roots of $\frac{dK}{ds}$ gives break away point

6) Intersection of root locus branches with the imaginary axis

Method 1

Find characteristic eqn

EXAMPLE: $1 + G_{OL}(s) = 0$

$$\Rightarrow s^3 + 6s^2 + 11s + (6+K) = 0$$

Substitute $s = j\omega$.

$$\Rightarrow -j\omega^3 - 6\omega^2 + 11j\omega + (6+K) = 0$$

$$-j\omega^3 + 11j\omega = 0 \quad 6+K - 6\omega^2 = 0$$

$$\Rightarrow \omega^2 = 11$$

$$\Rightarrow \omega = \pm\sqrt{11}$$

$$\Rightarrow K = -6 + 6\omega^2$$

$$= -6 + 6 \times 11$$

$$= -6 + 66$$

$$= 60$$

Method 2 : Find characteristic eqn. & Routh array

$$s^3 \rightarrow \begin{matrix} 1 \\ (s^3) \end{matrix} \quad \begin{matrix} 11 \\ (s^1) \end{matrix}$$

$$s^2 \rightarrow \begin{matrix} 6 \\ (s^2) \end{matrix} \quad \begin{matrix} 6+K \\ (s^0) \end{matrix}$$

$$s^1 \rightarrow \begin{matrix} 66-6-K \\ 6 \\ (s^1) \end{matrix} \quad 0$$

$$s^0 \rightarrow \begin{matrix} 6+K \\ (s^0) \end{matrix}$$

$$\frac{66-6-K}{6} = 0$$

$$\Rightarrow K = 60$$

$$6s^2 + 6 + K = 0$$

$$\Rightarrow 6s^2 + 66 = 0$$

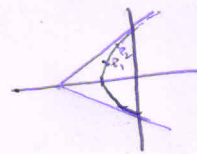
$$\Rightarrow s = \pm 11j$$

7) Angle of departure:

By trial & error.

Find P_1, P_2 on root locus or not

lets $P_1 = 2 + 3j$



root locus using Matlab

$$s = tf('s')$$

$$sys = 5*s/(s+3)*(s+5)$$

$$rlocus(sys)$$

$$\frac{5s}{(s+3)(s+5)}$$

$$num = [2 \ 0]$$

$$den = [1 \ 2 \ 3]$$

$$sys = tf(num, den) \left\} \frac{2s+0}{s^2+2s+3}\right.$$

$$rlocus(sys)$$

$$\Rightarrow \frac{2s}{s^2+2s+3}$$

$$num = [2]$$

$$den = [1 \ 2 \ 3]$$

$$sys = tf(num, den) \left\} \frac{2}{s^2+2s+3}\right.$$

$$rlocus(sys)$$

$$num = [1 \ 0];$$

$$den = [0 \ 1 \ 1];$$

$$sys = tf(num, den)$$

$$step(sys) \checkmark$$

$$[temp, t] = step(sys)$$

$$data = [t, temp]$$

$$step(num, den) \checkmark$$

$$z = 0$$

$$p = [1 \ 2 \ 3]$$

$$k = 5$$

$$\left\} \frac{5}{(s-1)(s-2)(s-3)}\right.$$

$$sys = zpkr(z, p, k)$$

$$rlocus(sys)$$

$$z = []$$

$$p = [1 \ 2 \ 3]$$

$$k = 5$$

$$\left\} \frac{5}{(s-1)(s-2)(s-3)}\right.$$

$$sys = zpkr(z, p, k)$$

$$rlocus(sys)$$

$$[r, k] = rlocus(sys, k)$$

$$r = rlocus(sys, k)$$

$$axis([-22 \ 3 \ -15 \ 15])$$

Frequency Response Analysis

The output of a system is sinusoidal when ~~the~~ a sinusoidal input is applied to it.

$$\text{Input} \Rightarrow A \sin(\omega t)$$

$$\text{output} \Rightarrow R \sin(\omega t + \phi)$$

$$\text{Amplitude ratio (AR)} = \frac{R}{A}$$

$$\text{Phase ^{Angle} lag} = \phi$$

Find AR & ϕ for different transfer function.

1. Substitute $j\omega$ ~~in~~ for s in $G(s)$

2. express $G(j\omega)$ in terms of $a + jb$

$$3. AR = \sqrt{a^2 + b^2}, \quad \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\Rightarrow G(s) = \frac{K_p}{\tau_p s + 1} \Rightarrow G(j\omega) = \frac{K_p}{\tau_p j\omega + 1}$$

$$G(j\omega) = \frac{K_p (\tau_p j\omega + 1)}{(\tau_p j\omega + 1)(\tau_p j\omega + 1)} = -\frac{K_p \tau_p \omega}{\tau_p^2 \omega^2 + 1} j + \frac{K_p}{\tau_p^2 \omega^2 + 1}$$

$$G(j\omega) = \frac{K_p}{(\tau_p \omega)^2 + 1} - j \frac{K_p \tau_p \omega}{(\tau_p \omega)^2 + 1}$$

$$AR = \sqrt{\frac{K_p^2 (1 + \tau_p^2 \omega^2)}{(\tau_p^2 \omega^2 + 1)^2}}$$

$$AR = \frac{K_p}{\sqrt{1 + \omega^2 \tau_p^2}}$$

$$\phi = \tan^{-1}(-\omega \tau_p)$$

$$\Rightarrow G(s) = \frac{K_p}{\tau_p^2 s^2 + 2\tau_p \zeta s + 1}$$

$$AR = \frac{K_p}{\sqrt{(-\tau_p^2 \omega^2)^2 + (2\tau_p \zeta \omega)^2}}$$

$$\phi = \tan^{-1}\left(\frac{-2\tau_p \zeta \omega}{1 - \tau_p^2 \omega^2}\right)$$

$$\Rightarrow G(s) = s, \quad G(j\omega) = j\omega$$

$$AR = \sqrt{0^2 + \omega^2} = \omega$$

$$\phi = \tan^{-1}\left(\frac{\omega}{0}\right) = \tan^{-1}(\infty) = 90^\circ$$

$$\Rightarrow G(s) = \frac{1}{s}, \quad G(j\omega) = \frac{1}{j\omega} = \frac{j}{j^2\omega} = -\frac{j}{\omega}$$

$$AR = \sqrt{0^2 + \left(\frac{1}{\omega}\right)^2} = \frac{1}{\omega}$$

$$\phi = \tan^{-1}\left(\frac{-1/\omega}{0}\right) = \tan^{-1}(-\infty) = -\tan^{-1}(\infty) = -90^\circ$$

$$\Rightarrow G(s) = K_P (\cancel{\tau_P s + 1}) K_P$$

$$AR = 1, \quad \phi = 0^\circ$$

$$\Rightarrow G(s) = K_P (\tau_P s + 1)$$

$$AR = \sqrt{1 + \tau_P^2 \omega^2}$$

$$\phi = \tan^{-1}(\omega \tau_P)$$

$$\Rightarrow G(s) = s^2$$

$$AR = \omega^2, \quad \phi = 180^\circ \quad [90 \times 2]$$

$$\Rightarrow G(s) = \frac{1}{s^2}$$

$$AR = \frac{1}{\omega^2}, \quad \phi = -180^\circ \quad [-90 \times 2]$$

$$AR = (AR)_1 \times (AR)_2 \times (AR)_3 \dots$$

$$\phi = \phi_1 + \phi_2 + \phi_3 + \dots$$

②

Frequency Response Analysis

$$M = 20 \log_{10} AR$$

Find M & ϕ for $\omega \ll \frac{1}{T_p}$ or very small value.

$\omega \gg \frac{1}{T_p}$ or very large value.

$\omega = \frac{1}{T_p}$ or at corner frequency

$$1. G(s) = s$$

$$AR = \omega$$

$$M = 20 \log_{10} AR$$

$$\phi = 90^\circ$$

~~AR value~~

ω	1	0.1	10
M	0	-20	20

20 db/decade

$$2. G(s) = \frac{1}{s}$$

$$AR = \frac{1}{\omega}$$

$$M = -20 \log_{10} AR$$

$$\phi = -90^\circ$$

ω	1	0.1	10
M	0	20	-20

-20 db/decade

$$3. G(s) = \frac{1}{0.1s + 1}$$

$$= \frac{1}{\frac{1}{10}s + 1}$$

$$AR = \frac{1}{\sqrt{\left(\frac{\omega}{10}\right)^2 + 1}}$$

$$M = -20 \log_{10} \sqrt{\left(\frac{\omega}{10}\right)^2 + 1}$$

$$\phi = -\tan^{-1}\left(\frac{\omega}{10}\right)$$

ω	10	1	100	0.1	1000	0.01
M	0	-20	0	-40	0	

ω	10	1	100	0.1	1000
ϕ	-45°	0	-90	0	-90

$$4. G(s) = 0.1s + 1$$

$$AR = \sqrt{\frac{\omega^2}{10^2} + 1}$$

$$\phi = \tan^{-1}\left(\frac{\omega}{10}\right)$$

$$m = 20 \log\left(\sqrt{\frac{\omega^2}{10^2} + 1}\right)$$

ω	10	0.1	0.01	100	1000
m		0	0	20	40

ω	10	0.1	0.01	100	1000
ϕ	45°	0°	0°	90°	90°

45°/decade

5.

$$s^2 = s \cdot s$$

$$AR = (AR)_1 \times (AR)_2$$

$$AR = \omega \times \omega = \omega^2$$

$$\phi = \phi_1 + \phi_2$$

$$= 90^\circ + 90^\circ$$

$$= 180^\circ$$

$$M = 20 \log_{10} AR$$

$$\cancel{40}$$

$$= 20 \log_{10} \omega^2$$

$$= 40 \log_{10} \omega$$

6.

$$\frac{1}{s^2} \Rightarrow \frac{1}{s} \cdot \frac{1}{s}$$

$$AR = (AR)_1 \times (AR)_2$$

$$AR = \frac{1}{\omega} \times \frac{1}{\omega} = \frac{1}{\omega^2}$$

$$\phi = \phi_1 + \phi_2$$

$$= -90^\circ - 90^\circ$$

$$= -180^\circ$$

$$M = 20 \log AR$$

$$\cancel{40}$$

$$= -40 \log \omega$$

③

Bode Plot

A Bode Plot is a useful tool that shows the gain and phase response of a given system frequencies.

Bode plot $\Rightarrow$ AR vs ω , ϕ vs ω
or
db

$$G(s) = 40 \frac{(1 + \frac{s}{10})}{s(1 + \frac{s}{50})(1 + \frac{s}{200})}$$

Final angle: $(\# \text{ poles} - \# \text{ zeros}) \times (-90^\circ)$
 $= (3 - 1) \times (-90^\circ)$
 $= -180^\circ$

6 Stability characteristic eqⁿ

$$1 + G(s) = 0$$

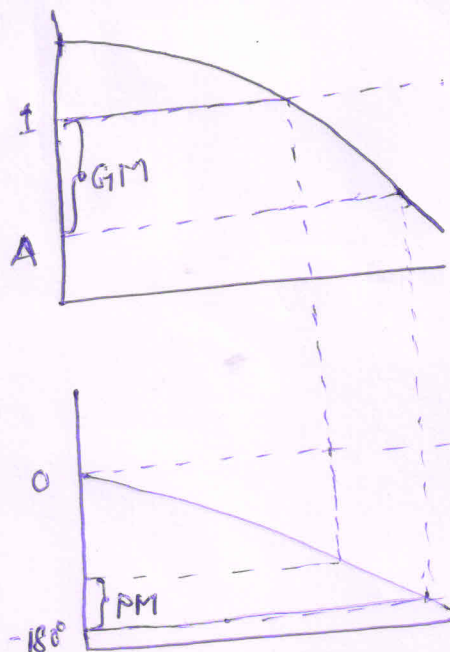
$$G(s) = -1$$

$$\text{Gain} = |G(s)| = 1 \Rightarrow 20 \log(1) = 0 \text{ db}$$

$$\text{phase: } \angle G(s) = \tan^{-1}\left(\frac{0}{-1}\right) = -180^\circ$$

If any frequency (ω) produces '0' db gain & 180° phase lag the closed loop system is unstable.

must be
phase $> -180^\circ$
Gain < 1



Gain margin $= 1/A$
 Gain margin > 1.7
 Phase margin $> 30^\circ$

Controller Tuning

Ziegler Nichols tuning

Step 1: Find AR & ϕ of the system.

Step 2: Find cross over frequency (ω_{co})

$\omega_{co} \Rightarrow$ frequency at which ϕ is -180° .

Step 3: Find AR at $\omega = \omega_{co}$, at $\omega = \omega_{co}$, $AR = A$

$$K_u = \frac{1}{A}, \quad P_u = \frac{2\pi}{\omega_{co}}$$

$K_u \Rightarrow$ Ultimate gain

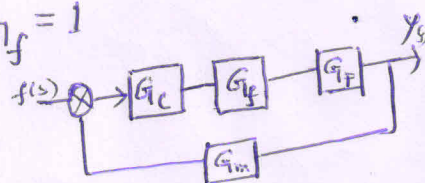
$P_u \Rightarrow$ Ultimate period of sustained cycling
(min/cycle)

Step 4: Find controller parameter for different controller from the following table.

	$G_c(s)$	K_c	τ_I	τ_D
P	K_c	$0.5 K_u$		
PI	$K_c(1 + \frac{1}{\tau_I s})$	$0.45 K_u$	$\frac{P_u}{1.2}$	
PID	$K_c(1 + \frac{1}{\tau_I s} + \tau_D s)$	$0.6 K_u$	$\frac{P_u}{2}$	$\frac{P_u}{8}$

Example:

$$G_p = \frac{1}{(5s+1)(2s+1)}, \quad G_m = \frac{1}{10s+1}, \quad G_f = 1$$



$$\phi = \tan^{-1}(-5\omega) + \tan^{-1}(-2\omega) + \tan^{-1}(-10\omega)$$

$$\log AR = \log \frac{1}{\sqrt{(5\omega)^2 + 1}} + \log \frac{1}{\sqrt{(2\omega)^2 + 1}} + \log \frac{1}{\sqrt{(10\omega)^2 + 1}}$$

$$G_{OL}(s) = G_f G_p G_m$$

$$\text{Find } \omega_{co} \rightarrow -180^\circ = -\tan^{-1} \left(\frac{5\omega_{co} + 2\omega_{co} + 10\omega_{co} - 5 \cdot 2 \cdot 10 \omega_{co}^3}{1 - 10\omega_{co}^2 - 20\omega_{co}^2 - 50\omega_{co}^2} \right)$$

$$\Rightarrow \frac{5\omega_{co} + 2\omega_{co} + 10\omega_{co} - 100\omega_{co}^3}{1 - 10\omega_{co}^2 - 20\omega_{co}^2 - 50\omega_{co}^2} = 0$$

$$5\omega_{co} + 2\omega_{co} + 10\omega_{co} - 100\omega_{co}^3 = 0$$

$$\Rightarrow 17\omega_{co} - 100\omega_{co}^3 = 0$$

$$\Rightarrow 17 - 100\omega_{co}^2 = 0$$

$$\Rightarrow \omega_{co}^2 = \frac{17}{100} = 0.17$$

$$\Rightarrow \omega_{co} = \sqrt{0.17}$$

$$\omega_{co} = 0.4123 \text{ rad/min}$$

$$\begin{aligned} AR|_{\omega_{co}} &= \frac{1}{\sqrt{25\omega_{co}^2 + 1}} \times \frac{1}{\sqrt{4\omega_{co}^2 + 1}} \times \frac{1}{\sqrt{100\omega_{co}^2 + 1}} \\ &= \frac{1}{\sqrt{25 \times 0.17 + 1}} \times \frac{1}{\sqrt{4 \times 0.17 + 1}} \times \frac{1}{\sqrt{100 \times 0.17 + 1}} \\ &= 0.0794 = A \end{aligned}$$

$$K_u = \frac{1}{A} = 12.6$$

$$P_u = \frac{2\pi}{\omega_{co}} = 15.239 \text{ min/cycle}$$

Now, from table.

Proportional Controller: $K_c = \frac{12.6}{2} = 6.3$

PI Controller: $K_c = 0.45 \times 12.6 = 5.7$

$$\tau_I = \frac{15.239}{1.2} = 12.62 \text{ min}$$

PID Controller: $K_c = 0.6 \times 12.6 = 7.4$

$$\tau_I = \frac{15.239}{2} = 7.57 \text{ min}$$

$$\tau_D = \frac{15.239}{8} = 1.89 \text{ min}$$

Example:

$$G_p = \frac{1}{s+1}, G_m = 1, G_f = 1, G_T = e^{-1.02s}$$

$$\hookrightarrow AR = 1$$

$$\phi = -57.3(1.02)\omega$$

$$\omega_{co} \rightarrow -180^\circ = -57.3 \times 1.02 \omega_{co}$$

$$\Rightarrow \omega_{co} = 2 \text{ rad/min} \quad AR|_{\omega_{co}} = A = \frac{1}{2.24}$$

$$K_u = 2.24, P_u = \frac{2\pi}{\omega_{co}} = 3.142 \text{ min/cycle}$$

$$5\omega_{co} + 2\omega_{co} + 10\omega_{co} - 100\omega_{co}^3 = 0$$

$$\Rightarrow 17\omega_{co} - 100\omega_{co}^3 = 0$$

$$\Rightarrow 17 - 100\omega_{co}^2 = 0$$

$$\Rightarrow \omega_{co}^2 = \frac{17}{100} = 0.17$$

$$\Rightarrow \omega_{co} = \sqrt{0.17}$$

$$\omega_{co} = 0.4123 \text{ rad/min}$$

$$\begin{aligned} AR|_{\omega_{co}} &= \frac{1}{\sqrt{25\omega_{co}^2 + 1}} \times \frac{1}{\sqrt{4\omega_{co}^2 + 1}} \times \frac{1}{\sqrt{100\omega_{co}^2 + 1}} \\ &= \frac{1}{\sqrt{25 \times 0.17 + 1}} \times \frac{1}{\sqrt{4 \times 0.17 + 1}} \times \frac{1}{\sqrt{100 \times 0.17 + 1}} \\ &= 0.0794 = A \end{aligned}$$

$$K_u = \frac{1}{A} = 12.6$$

$$P_u = \frac{2\pi}{\omega_{co}} = 15.239 \text{ min/cycle}$$

Now, from table.

$$\text{Proportional Controller: } K_c = \frac{12.6}{2} = 6.3$$

$$\text{PI Controller: } K_c = 0.45 \times 12.6 = 5.7$$

$$\tau_I = \frac{15.239}{1.2} = 12.62 \text{ min}$$

$$\text{PID Controller: } K_c = 0.6 \times 12.6 = 7.4$$

$$\tau_I = \frac{15.239}{2} = 7.57 \text{ min}$$

$$\tau_D = \frac{15.239}{8} = 1.89 \text{ min}$$

Example:

$$G_P = \frac{1}{s+1}, G_m = 1, G_f = 1, G_T = e^{-1.02s}$$

$$\hookrightarrow AR = 1$$

$$\phi = -57.3(1.02)\omega$$

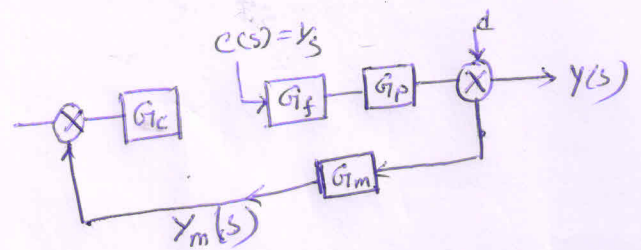
$$\omega_{co} \rightarrow -180^\circ = -\tan^{-1}(\omega_{co}) - 57.3 \times 1.02 \omega_{co}$$

$$\Rightarrow \omega_{co} = 2 \text{ rad/min}, AR|_{\omega_{co}} = A = \frac{1}{2.24}$$

$$K_u = 2.24, P_u = \frac{2\pi}{\omega_{co}} = 3.142 \text{ rad/min} \frac{\text{min}}{\text{cycle}}$$

Controller Tuning

Process reaction curve method by Cohen & Coon



Step 1: * The response of most processing units to an step input change, have a sigmoidal shape.

* The response can be adequately approximated by the response of a first-order system with dead time.

Step 1: Loop opened by disconnecting controller from FCE.

Step 2: Introduce step input in $c(s) = A/s$

Step 3: Record value of output $y_m(t)$

Step 4: Plot $y_m(t)$ vs t . curve $y_m(t)$ vs t is called process reaction curve (PRC) and it is a sigmoidal curve.

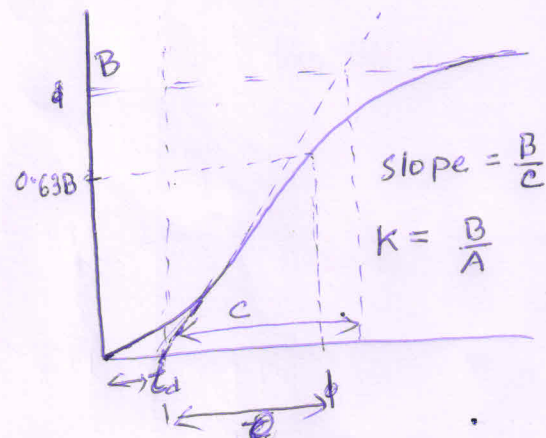
Step 5: Express curve in terms of first order ~~transfer~~ ^{with dead time} form.

find t_d , K , τ

$$G_{IPRC}(s) = \frac{y_m(s)}{c(s)} = \frac{K}{\tau s + 1} e^{-t_d s}$$

$$\tau = \frac{B}{S} \left[\because \text{with initial slope at } t = \tau, \text{ system reaches final value} \right]$$

t_d = time elapsed until the system responded



Step 6: use these approximate model of first order with dead time and estimate the value of controller parameters.

a. P $\Rightarrow K_c = \frac{1}{K} \frac{\tau}{t_d} \left(1 + \frac{t_d}{3\tau} \right)$

b. PI $\Rightarrow K_c = \frac{1}{K} \frac{\tau}{t_d} \left(0.9 + \frac{t_d}{12\tau} \right)$; $\tau_I = t_d \left(\frac{30 + 3t_d/\tau}{9 + 20t_d/\tau} \right)$

c. PID $\Rightarrow K_c = \frac{1}{K} \frac{\tau}{t_d} \left(\frac{4}{3} + \frac{t_d}{4\tau} \right)$; $\tau_I = t_d \left(\frac{32 + 6t_d/\tau}{13 + 8t_d/\tau} \right)$; $\tau_D = \frac{4t_d}{11 + 2t_d/\tau}$

Controller design using different performance criteria & tuning of the controller

Q1. What type of feedback controller should be used to control a given process?

⇒ P ⇒ offset is acceptable.

PI ⇒ no offset desirable & more overshoot & settling time acceptable.

~~PID~~ ⇒ PD ⇒ offset is acceptable but less overshoot & settling time desirable.

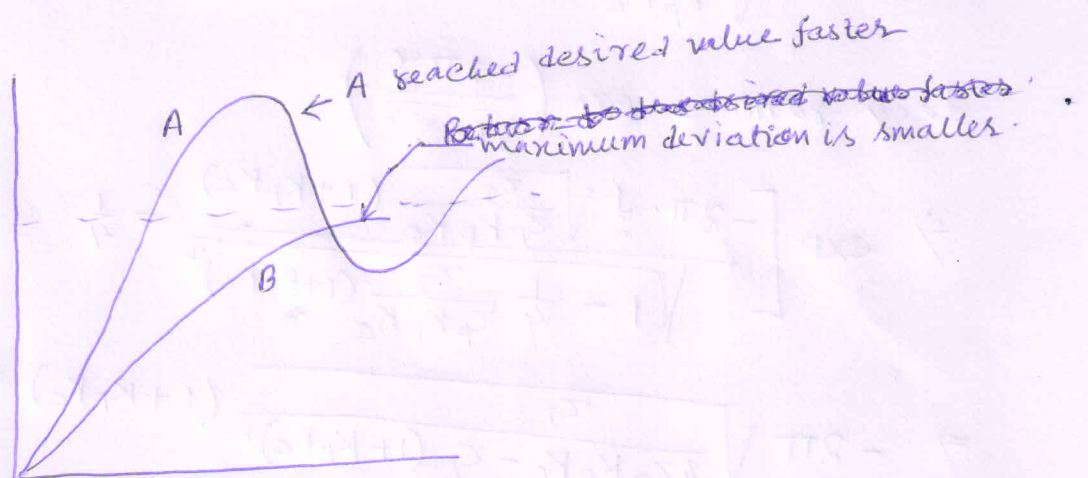
PID ⇒ no offset & less overshoot & settling time desirable.

Q2. How do we select the best values for the adjustable parameters of a feedback controller?

⇒ Controller tuning.

Q3. What performance criteria should we use for the selection and the tuning of the controller?

- ^{max} Maximum deviation ^(error) as small as possible
- Achieve short settling time.
- Minimise the integral of the error until the process value reaches steady state ^{desired} point.
- etc



Types of performance criteria

1. Steady state performance criteria
2. Dynamic response performance criteria

1. The Principal steady state performance criterion usually is zero error at steady state.

PI $\Rightarrow$ zero error

P $\Rightarrow$ @ $K \rightarrow \infty$, zero error.

2. Dynamic response performance criteria

a. criteria uses only few points of the response
- approximate.

b. criteria uses entire close loop response for long time.
- more precise.
(Time integral performance criteria)

a. i) minimum overshoot

ii) minimum settling time

iii) minimum rise time

iv) $1/4$ th decay ratio (most popular)

Example!

one-quarter
decay ratio

$$Y(s) = \frac{(\tau_I s + 1)}{\tau_I^2 s^2 + 2\tau_I s + 1} \bar{y}_{sp}(s)$$

$$\tau = \sqrt{\frac{\tau_I \tau_p}{K_p K_c}}$$

$$\zeta = \frac{1}{2} \sqrt{\frac{\tau_I}{\tau_p K_p K_c}} (1 + K_p K_c)$$

$$\text{decay ratio} = \exp\left(\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$\Rightarrow \exp\left[\frac{-2\pi \cdot \frac{1}{2} \sqrt{\frac{\tau_I}{\tau_p K_p K_c}} (1 + K_p K_c)}{\sqrt{1 - \frac{1}{4} \frac{\tau_I}{\tau_p K_p K_c} (1 + K_p K_c)^2}}\right] = \frac{1}{4}$$

$$\Rightarrow -2\pi \sqrt{\frac{\tau_I}{4\tau_p K_p K_c - \tau_I (1 + K_p K_c)^2}} (1 + K_p K_c) = \ln(1/4)$$

select different values of K_c & get corresponding τ_I value.

$$K_c = 1 \\ \tau_I = 0.153$$

$$K_c = 10 \\ \tau_I = 0.464$$

$$K_c = 30 \\ \tau_I = 0.348$$

$$K_c = 100 \\ \tau_I = 0.153$$

Time-Integral performance criteria

i) Integral of the square error (ISE)

$$ISE = \int_0^{\infty} e^2(t) dt$$

minimise
ii) Integral of the absolute value of the error (IAE)

$$IAE = \int_0^{\infty} |e(t)| dt$$

iii) Integral of the time-weighted absolute error (ITAE)

$$ITAE = \int_0^{\infty} t |e(t)| dt$$

$$e = y_{sp}(t) - y(t)$$

minimise the ISE, IAE or ITAE wrt K_c & τ_I & τ_D to adjust controller parameters

* If we want to strongly suppress large error, ISE is better than IAE, because the errors are squared and thus contribute more to the value of the integral.

* For the suppression of small errors, IAE is better than ISE, because when small numbers (0.1) they become even smaller.

* To suppress error that persist for long times the ITAE gives better result, because the presence of large t amplifies the effect of even small errors in the value of the integral.

Step 1

Express system response ~~in terms of~~ as a function of controller parameters by ~~un~~ knowing other process parameters

$$Y(s) = \frac{G_c G_f G_p}{1 + G_c G_f G_p G_m} \bar{y}_{sp}(s) + \frac{G_d}{1 + G_c G_f G_p G_m} \bar{d}(s)$$

$$Y(s) = \frac{\tau_I s + 1}{\frac{\tau_I}{20K_c} s^2 + \tau_I \left(1 + \frac{1}{20K_c}\right) s + 1} \bar{y}_{sp}(s) + \frac{(\tau_I / 20K_c) s}{\frac{\tau_I}{20K_c} s^2 + \tau_I \left(1 + \frac{1}{20K_c}\right) s + 1} \bar{d}(s)$$

$$Y(s) = \frac{\tau_I s + 1}{\tau_I^2 s^2 + 2\tau_I s + 1} \bar{y}_{sp}(s) + \frac{(\tau_I / 20K_c) s}{\tau_I^2 s^2 + 2\tau_I s + 1} \bar{d}(s)$$

$$\tau = \sqrt{\frac{\tau_I}{20K_c}} ; \zeta = \frac{1}{2} \sqrt{\frac{\tau_I}{20K_c}} (1 + 20K_c)$$

Instead of taking derivative wrt K_c, τ_I we can take derivative wrt τ & ζ

Step 2: Find $\bar{y}(s)$ for a particular change in $\bar{y}_{sp}(s)$ or $\bar{d}(s)$ or both.

Step 3: Find error $e = \bar{y}_{sp} - \bar{y}(s)$

Step 4: Find, ISE, IAE or ITAE

Step 5: Minimise ISE, IAE or ITAE

$$\frac{\partial (ISE)}{\partial \tau} = 0, \frac{\partial (ISE)}{\partial \zeta} = 0 \quad \left[\begin{array}{l} \text{Get two eqn, solve} \\ \text{to find } \tau \text{ \& } \zeta \Rightarrow \text{then find} \\ K_c \text{ \& } \tau_I \\ \text{as } K_c \text{ \& } \tau_I \text{ is function} \\ \text{of } \tau \text{ \& } \zeta \end{array} \right]$$

$$\text{or } \frac{\partial (IAE)}{\partial \tau} = 0, \frac{\partial (IAE)}{\partial \zeta} = 0$$

$$\text{or } \frac{\partial (ITAE)}{\partial \tau} = 0, \frac{\partial (ITAE)}{\partial \zeta} = 0$$

$$\tau = \sqrt{\frac{\tau_I}{20K_c}}$$

$$\zeta = \frac{1}{2} \sqrt{\frac{\tau_I}{20K_c}} (1 + 20K_c)$$